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On the Burnside Problem for Semigroups

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The Burnside problem for semigroups is the following: Given a finitely generated semigroup S, each element of which generates a finite subsemigroup (i.e., S is torsion), is S finite?

This problem has a negative answer in general (Morse and Hedlund, see [3]). In fact this problem was first raised for groups, for which the answer is also negative (Golod and Shafarevitch, see [1, Chap. 8]). The answer is trivially positive for commutative semigroups. In this note we introduce a property of semigroups, *the permutation property*, which generalizes commutativity; we show that each finite semigroup has this property and that the Burnside problem has a positive answer for semigroups having this property: this is a consequence of a combinatorial theorem, due to Shirshov.

Let S be a semigroup. We say that S has the permutation property if there exists some integer $n \geqslant 2$ such that, for any elements $s_1, s_2, \ldots, s_n$ in S, there is some permutation σ of $\{1, 2, \ldots, n\}$, $\sigma \neq$ id., verifying

$$s_1 \cdots s_n = s_{\sigma(1)} \cdots s_{\sigma(n)}.$$

Note that for $n = 2$, this is commutativity.

THEOREM. *Let S be a finitely generated semigroup. The following conditions are equivalent*:

(i) *S is finite.*

(ii) *S is torsion and has the permutation property.*

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Proof. (i) $\Rightarrow$ (ii). Let $n = 1 + 2\,\mathrm{Card}(S)$ and $s_1, \ldots, s_n \in S$. Consider the sequence

$$s_1, s_1 s_2, \ldots, s_1 \cdots s_n$$

of n elements of S. By definition of n, there exist i, j, k, $1 \leqslant i < j < k \leqslant n$, such that

$$s_1 \cdots s_i = s_1 \cdots s_j = s_1 \cdots s_k.$$

Let $u = s_1 \cdots s_i$, $x = s_{i+1} \cdots s_j$, $y = s_{j+1} \cdots s_k$. Then $u = ux = uxy$. Hence

$$\begin{aligned} uyx &= uxyx && (\text{because } u = ux) \\ &= ux && (\text{because } uxy = u) \\ &= uxy. \end{aligned}$$

Thus $s_1 \cdots s_k = s_1 \cdots s_i s_{j+1} \cdots s_k s_{i+1} \cdots s_j$ and this shows that S has the permutation property.

(ii) $\Rightarrow$ (*i*). Let S have the permutation property for n. Let A be a finite set generating S. Let A^+ be the free semigroup generated by A and $\varphi: A^+ \to S$ the natural semigroup homomorphism; φ is surjective. An element w of A^+ is called word and $|w|$ is its length. Let $<$ be a total order on A. We order A^+: for $u, v \in A^+$, $u < v$ if and only if either $|u| < |v|$, or $|u| = |v|$ and u is smaller than v with respect to the lexicographical ordering. A word w is called n-divided if it admits some factorization

$$w = ux_1 \cdots x_n v$$

such that for each permutation σ of $\{1, \ldots, n\}$, $\sigma \neq$ id., one has

$$w > ux_{\sigma(1)} \cdots x_{\sigma(n)} v. \tag{1}$$

We say that a word w contains a pth power of a word x, if, for some words u, v, w may be written $w = ux^p v$. By a theorem of Shirshov (see [4, Theorem 4.2.7] or [2, Chap. 8]) there exists for each $p \geqslant 2n$, an integer $N(p)$ such that each word of length at least $N(p)$ either is n-divided or contains a pth power of a word of length at most $n - 1$.

Choose $p \geqslant 2n$ such that for each word w of length less than n, the element $\varphi(w)^p$ of S is equal to $\varphi(w)^{p'}$ for some $p' < p$. This is possible because S is torsion and A finite. We show that, for each s in S, $s = \varphi(w)$ for some word w with $|w| < N(p)$ (hence S is finite). Indeed, let w be minimum in the subset $\varphi^{-1}(s)$ of A^+. Then $|w| < N(p)$: suppose that it is not the case. Then, by Shirshov's theorem, either w is n-divided, or w contains a pth power of x, with $|x| < n$.

In the first case, $w = ux_1 \cdots x_n v$: By the permutation property, there is some $\sigma \neq$ id. such that

$$s = \varphi(ux_{\sigma(1)} \cdots x_{\sigma(n)} v).$$

But, by (1), $ux_{\sigma(1)} \cdots x_{\sigma(n)} v < w$ and this contradicts the definition of w.

In the second case, $w = ux^p v$: By definition of p, one has

$$s = \varphi(ux^{p'} v)$$

with $p' < p$ and this is a contradiction, too.

Remarks. 1. Let $A = \{a_1, \ldots, a_{n-1}\}$ and S be the quotient of A^+ by the ideal of all words of length at least n. Then S has the permutation property for n, but not for $n - 1$.

2. There is a connection with rings with polynomial identities. Indeed, if for some commutative ring k, the k-algebra $k[S]$ of S satisfies a polynomial identity, then it satisfies a multilinear identity (see [4, Chap. 1]) and this implies easily that S has the permutation property. By the way, if S is finite, then S satisfies the standard identity of degree Card$(S) + 1$ (see [4]), hence S has the permutation property for $n = \text{Card}(S) + 1$; this proves again (i) $\Rightarrow$ (ii), with a smaller n.

The previous remark raises the problem whether, for each semigroup S having the permutation property, $k[S]$ satisfies a polynomial identity.

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