Europ. J. Combinatorics (1990) **11**, 501–512

Combinatorial Resolution of Systems of Differential Equations III: a Special Class of Differentially Algebraic Series

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We introduce and give a combinatorial model for a new class of formal power series: constructible differentially algebraic series. We show that this class is an algebra which is closed for inversion, substitution and inverse for substitution and study properties of their coefficients. We compare it to other families of series and give many examples and counter-examples (involving Euler, Bell, Stirling and Genocchi numbers).

1. INTRODUCTION

A series x in $\mathbb{C}[[t]]$ is called *differentially algebraic* if for some non-zero polynomial P in $k+2$ variables over $\mathbb{C}$, one has $P(t, x, x', \ldots, x^{(k)}) = 0$. Many classical analytic functions are differentially algebraic. These series have been studied by several authors (see, e.g., [5], [15], [17], [18]).

We introduce here a special class of differentially algebraic series, which we call *constructible differentially algebraic series* (CDF). A series x in $\mathbb{C}[[t]]$ is CDF if for some $k \geqslant 1$, there exist k series $x_1, \ldots, x_k$ with $x_1 = x$ and polynomials $P_1, \ldots, P_k$ such that

$$x_1' = P_1(x_1, \ldots, x_k),$$
$$\vdots$$
$$x_k' = P_k(x_1, \ldots, x_k).$$

We shall show in Section 5 that such a series is also differentially algebraic.

First, we give a combinatorial interpretation of constructible differentially algebraic series in terms of increasing trees, a well known object of combinatorics. This approach is closely related to the work of Leroux-Viennot [12–14] where the authors use species of M-enriched increasing arborescences to solve combinatorially differential equations. However, we shall not explicitly use the theory of species; we need just a combinatorial Gröbner-like formula, and our combinatorial interpretation of the powers of a linear differential operator [1].

In Section 3 many examples of classical generating functions are shown to be CDF. In Section 4, we proceed with the study of CDF series from a point of view inspired by Stanley's treatment of D-finite series [19]. We show in this light that the class of CDF series has all the usual closure properties, except for the Hadamard product. We further establish some properties of the coefficients of these series, in particular a kind of Fatou property (as studied in [2], chapter 5). Finally, in Section 5, we compare the family of CDF series to other known families: differential algebraic series, algebraic series and D-finite series, series satisfying an equation of the form:

$$x^{(k)} = P(x, x', \ldots, x^{(k-1)}).$$

This paper is the third of a series originated by Leroux-Viennot [12–14] on the combinatorial resolution of systems of differential equations. One objective of this series is to show that a combinatorial outlook may give a new insight into the resolution of systems of differential equation; and conversely, some combinatorial problems may be solved by differential equations.

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2. Constructible Differentially Algebraic Series

We say that x in $\mathbb{C}[\![t]\!]$ is a *constructible differentially algebraic series* (CDF series, for short; we borrow the word 'constructible' from [6]) if there exist k series $x_1, \ldots, x_k$ in $\mathbb{C}[\![t]\!]$ with $x_1 = x$, and k polynomials $P_1(X_1, \ldots, X_k)$, $P_2(X_1, \ldots, X_k), \ldots, P_k(X_1, \ldots, X_k)$ such that

$$\begin{aligned} x_1' &= P_1(x_1, \ldots, x_k), \\ x_2' &= P_2(x_1, \ldots, x_k), \\ &\vdots \\ x_k' &= P_k(x_1, \ldots, x_k). \end{aligned} \tag{1}$$

For example, the series $x = \sec(t)$ is CDF: let $x_1 = x$ and $x_2 = \operatorname{tg}(t)$; then

$$x_1' = x_1 x_2, \qquad x_2' = 1 + x_2^2. \tag{2}$$

In order to give a combinatorial interpretation for the solutions of system (1), we consider trees (rooted non-ordered trees) such as shown in Figure 1, which are defined by the following conditions:
(i) The set of vertices is a totally ordered seet, usually $[n] = \{1, 2, \ldots, n\}$ with the traditional order.
(ii) The tree is *increasing,* meaning that each son of each vertex is greater than the latter.
(iii) The vertices may have different 'colors' (here □ and ○).
(iv) We do not distinguish between the tree shown in Figure 1 and that shown in Figure 2.

We shall give weights to such trees in the following manner. Let Γ be the set of colors for the vertices. A *tree labelling table* is a function from Γ to the ring $\mathbb{C}[\Gamma]$ of polynomials in the variables Γ. As an example, let $\alpha \in \mathbb{C}$; then we might associate to each of the two colors □ and ○, the polynomials:

$$\begin{aligned} &\square \to \alpha + \alpha\square + \alpha\bigcirc + \alpha\bigcirc\square, \\ &\bigcirc \to 1 + \bigcirc. \end{aligned} \tag{3}$$

The *weight* of a colored increasing tree T is the product of the weights of all its vertices. The weight of a vertex v of T of color C, with n_i sons of color C_i, is the coefficient of the monomial $\Pi_i C_i^{n_i}$ in the polynomial corresponding to the color C, as given by the tree labelling table. The generating series for such trees, with a given color for the root, will be

$$x(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!},$$

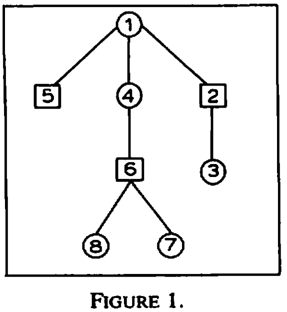

Figure 1.

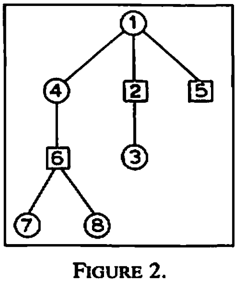

Figure 2.

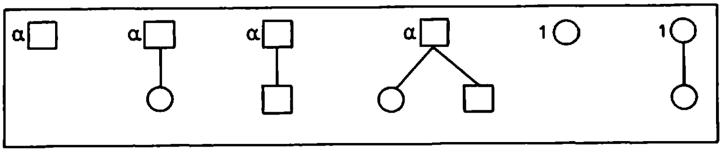

FIGURE 3.

where a_n is equal to the sum of the weights of all increasing trees on $[n]$ generated by the given table and having a root of the chosen color. Note that a_0 is necessarily 0; thus series generated by tree labelling tables have zero as constant term. If any vertex of T has weight 0, then the weight of T is zero. It is clear that trees of weight zero on $\{1, 2, 3, \ldots, n\}$ do not contribute to a_n in the generating series. Hence they are excluded from consideration, and we might as well say that a tree T is *generated* by a tree labelling table if all its vertices have non-zero weight. This suggests the representation of Figure 3, for the patterns of adjacencies corresponding to table (3).

The trees on $[3] = \{1, 2, 3\}$, generated by table (3) and with root □, are as shown in Figure 4. Thus the coefficient of $t^3/3!$ in the series generated by (3) with color □ is $\alpha^3 + 3\alpha^2 + \alpha$.

THEOREM 1. *A series x is CDF iff $x - x(0)$ is generated by some tree labelling table.*

It would be easy to adapt the proof of Theorem 3.1 of Leroux-Viennot [12] to obtain a purely combinatorial proof of Theorem 1. But we have chosen to give a proof that is closer to the spirit of this paper.

PROOF OF THEOREM 1. The proof is in two parts. The first part follows Gröbner's method for the solution of systems of differential equations. Part two establishes, in the light of part 1, the bijective correspondence between tree labelling tables, and systems of form (1).

Part 1. Consider system (1) and let α_i be the constant term of x_i. Let $y_1, \ldots, y_k$ be new variables and let Δ be the linear differential operator of $\mathbb{C}[y_1, \ldots, y_k]$ defined by

$$\Delta = \sum_{i=1}^{k} P_i(y_1, \ldots, y_k) \frac{\partial}{\partial y_i}.$$

Then

$$x_i = \exp(t\Delta) y_i \Big|_{y_1=\alpha_1, \ldots, y_k=\alpha_k}; \tag{4}$$

or, equivalently,

$$x_i(t) = \sum_{n=0}^{\infty} [\Delta^n(y_i)\,|_{y=\alpha}] \frac{t^n}{n!} \tag{5}$$

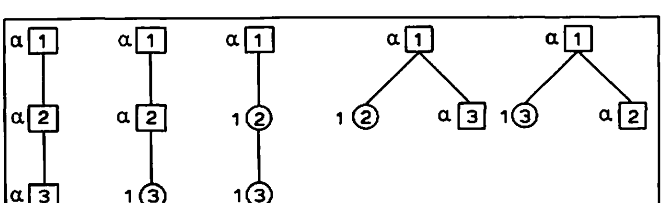

FIGURE 4.

(where $y=\alpha$ means $y_1=\alpha_1, \ldots, y_k=\alpha_k$). Indeed, the constant term of the right-hand side of (4) or (5) is α_i, as required. Moreover, as shown by Gröbner [9] (see also [7] or [11]), the mapping $\mathbb{C}[y_1, \ldots, y_k] \to \mathbb{C}[\![t]\!]$ that sends a polynomial $z(y_1, \ldots, y_k)$ to the series $\exp(t\Delta)z|_{y=\alpha}$ is an homomorphism of $\mathbb{C}$-algebras. Hence

$$\begin{aligned}\frac{d}{dt}[\exp(t\Delta)y_i\,|_{y=\alpha}] &= \exp(t\Delta)\,\Delta y_i\,|_{y=\alpha}\\ &= \exp(t\Delta)P_i(y_1, \ldots, y_k)\,|_{y=\alpha}\\ &= P_i(\exp(t\Delta)y_1\,|_{y=\alpha}, \ldots, \exp(t\Delta)y_k\,|_{y=\alpha}),\end{aligned}$$

and the series which are on the right-hand sides of (5) for $i=1, \ldots, k$ satisfy equation (1). The last equality follows from Gröbner's homomorphism.

Part 2. We now have to evaluate Δ^n. We may write

$$\Delta^n = \sum_{i=1}^{k} A_{i,n}\frac{\partial}{\partial y_i} + \Omega_n,$$

where the $A_{i,n}$'s are polynomials in $P_1, \ldots, P_k$ and their partial derivatives with respect to $y_1, \ldots, y_k$ and where Ω_n is a linear combination over $\mathbb{C}[y_1, \ldots, y_k]$ of derivations of order at least 2. Morover, a simple computation shows that

$$A_{i,n+1} = \sum_{j=1}^{k} P_j\frac{\partial A_{i,n}}{\partial y_j}.$$

This equation shows by induction on n that $A_{i,n}$ may be obtained in the following way (see [1] for details): $A_{i,n}$ is the sum of the weights of all increasing trees on $[n]$ with k colors $C_1, \ldots, C_k$ and having a root of colour C_i. The weight of a tree is the product of the weights of its nodes, and the weight of a node v of color C_h is

$$\left[\prod_{j=1}^{k}\frac{\partial^{k_j}}{\partial y_j^{k_j}}\right]P_h$$

when v has k_j sons of color C_j. By equation (5), it is enough to apply to the weights the substitutions $y_1 \to \alpha_1, \ldots, y_k \to \alpha_k$, to show that the series $x_1-\alpha_1, \ldots, x_k-\alpha_k$ are generated by the tree labelling table $\mathcal{J}$ on k colors $C_1, \ldots, C_k$, described below:

$$C_i \to \sum_{j_1,\ldots,j_k} \frac{\partial^{j_1+\cdots+j_k}}{\partial y_1^{j_1}\cdots\partial y_k^{j_k}}P_i\Big|_{y_1=\alpha_1,\ldots,y_k=\alpha_k} C_1^{j_1}\cdots C_k^{j_k}.$$

The above argument may be reversed. Hence if $\mathcal{J}$ is a tree labelling table with k colors $C_1, \ldots, C_k$ and corresponding polynomials

$$C_i \to \sum_{j_1,\ldots,j_k} b^{(i)}_{j_1,\ldots,j_k} C_1^{j_1}\cdots C_k^{j_k}, \tag{6}$$

then the series $x_1, \ldots, x_k$ generated by $\mathcal{J}$ satisfy equation (1), with

$$P_i = \sum_{j_1,\ldots,j_k} b^{(i)}_{j_1,\ldots,j_k}\frac{y_1^{j_1}\cdots y_k^{j_k}}{j_1!\cdots j_k!}. \tag{7}$$

□

The proof of Theorem 1 shows how to establish a correspondence between systems of equations such as (1), and tree labelling tables. If the table is given in form (6), the corresponding system is (7). Conversely, first replace each function x_i in (1) by $y_i = x_i - x_i(0)$, and in the system satisfied by the y_i, replace each monomials

$$y_1^{j_1} y_2^{j_2} \cdots y_k^{j_k}$$

by

$$j_1!\, j_2! \cdots j_k!\; C_1^{j_1} C_2^{j_2} \cdots C_k^{j_k}.$$

3. Some Examples

Example 1. Consider the series $x_1 = \exp(\alpha(e^t - 1))$ and $x_2 = e^t$. Then one has

$$x_1' = \alpha x_1 x_2, \qquad x_2' = x_2.$$

This shows (cf. equation (5)) that x_1 and x_2 are generated by table (3). Taking $\alpha = 1$, x_1 becomes the classical exponential generating function of the Bell numbers: hence, the nth Bell number is equal to the number of increasing trees on $[n]$, with root of color '□', and nodes either of color '□' or '○', where the adjacencies are given by Figure 3. It is not difficult to see directly that these trees count the number of partitions of a set of n elements, which is the classical property of Bell numbers. The tree representation of the partition $\{\{1,3,7\}, \{2,4\}, \{5\}, \{6,8\}\}$ of the set $\{1,2,3,4,5,6,7,8\}$ is shown in Figure 5.

Example 2. Now, consider the table $C \to 1 + 2C^2$. From equation (6), it follows that the series x generated by this table satisfies $x' = 1 + x^2$, hence $x = \mathrm{tg}(t)$. Recall that

$$\mathrm{tg}(t) = \sum_{n=0}^{\infty} E_{2n+1} \frac{t^{2n+1}}{(2n+1)!},$$

where E_n is the nth Euler number (see [20]). The trees generated by this table are the complete binary increasing trees. Any such tree with $2n+1$ nodes has n internal nodes and $n+1$ leaves. Thus, its weight is 2^n and we conclude (as in [20]) that the $(2n+1)$th Euler number E_{2n+1} is equal to 2^n times the number of complete increasing trees on $[2n+1]$. All possible such trees on the set $\{1,2,3,4,5\}$ are represented in Figure 6.

Example 3. Let $x = (1-t)^{-1}$, $x_k = (\log(1-t)^{-1})^k/k!$. We clearly have $x' = x^2$, $x_1' = x$ and $x_k' = x_{k-1}x$ (for $k \geq 2$). Thus, these series are CDF. Let '○' and '□j' be the colors corresponding respectively to x, and the x_j's. The corresponding adjacencies table is represented in Figure 7.

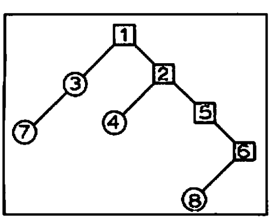

Figure 5.

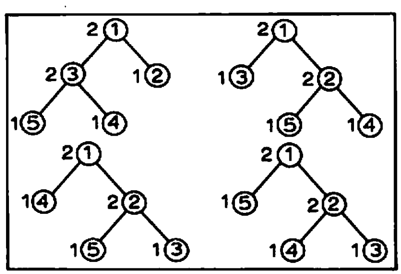

Figure 6.

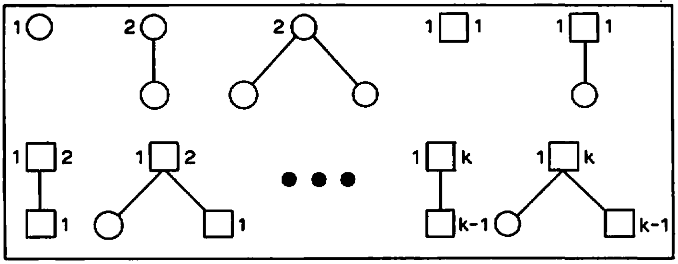

FIGURE 7.

Now, x is the exponential generating function of permutations, and the x_k's are the exponential generating function for the kth Stirling numbers (of the first kind). This shows, through Foata's transformation (see [8, p. 92]), the well known result that the kth Stirling number is the number of permutations having k cycles in their cycle decomposition.

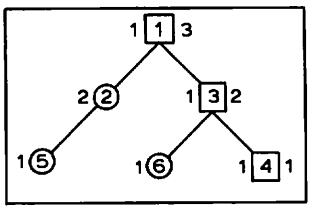

FIGURE 8. Tree of weight two counting the two permutations (125) (36) (4) and (152) (36) (4) of [6].

4. Closure Properties

We shall see that the set of CDF series has all the usual closure properties.

THEOREM 2. *The set of CDF series is a subalgebra of* $\mathbb{C}[[t]]$ *which contains* $\mathbb{C}[t]$ *and which is closed for inversion, composition, inversion for composition and integration.*

As the proof will show, all these closure properties are *effective*: given a finite set of CDF series (by systems such as equation (1), or tables), one may effectively construct a system or a table corresponding to a series obtained from the above series by a finite number of the operations mentioned in the theorem. We first establish a simple but useful lemma.

LEMMA 1. *A series is CDF if and only if it is contained in some finitely generated subalgebra of* $\mathbb{C}[[t]]$ *which is closed for differentiation.*

Note that if a subalgebra of $\mathbb{C}[[t]]$ is generated by a set X of series and contains x' for any x in X, then it is closed for differentiation.

PROOF. If equation (1) is satisfied, then the subalgebra of $\mathbb{C}[[t]]$ generated by $x_1, \ldots, x_k$ is closed under differentiation. It is finitely generated and contains x. Conversely, suppose that x is contained in a subalgebra A of $\mathbb{C}[[t]]$ which is finitely generated and closed for differentiation. Let $x_1, \ldots, x_k$ generate A, with $x_1 = x$; then x_i' is in A. Thus $x_i' = P_i(x_1, \ldots, x_k)$ for polynomials P_i in the generators, and equation (1) is satisfied. □

PROOF OF THEOREM 2. We say *DF algebra* for finitely generated subalgebras of $\mathbb{C}[[t]]$ closed under differentiation.

1. $\mathbb{C}[t]$ is a DF algebra, hence any polynomial is CDF, by Lemma 1.

2. Let x and y be CDF series. By Lemma 1, they are contained in two DF algebras A and B, respectively. The algebra generated by $A \cup B$ is a *DF* algebra and contains x', $x + y$ and xy. Hence these series are also CDF.

3. Suppose x' is CDF. Then it is contained in some DF algebra A. The algebra generated by A and x is a DF algebra; hence x is CDF.

4. Let x be invertible and contained in some DF algebra A. Then the algebra generated by A and x^{-1} is a DF algebra since $(x^{-1})' = -x'x^{-2}$; hence x^{-1} is CDF.

5. Let x and y be in DF algebras A and B, respectively. Suppose $x(0) = 0$, and let $x_1, \ldots, x_k$ generate A and $y_1, \ldots, y_m$ generate B. Let C be the algebra generated by $x_1, \ldots, x_k, y_1(x), \ldots, y_m(x)$. Then C is closed for differentiation, because of the chain rule $(y_i(x))' = y_i'(x) \cdot x'$. Hence C is a DF algebra containing $y(x)$, which is therefore CDF.

6. Let $x = \sum_{n\geq 0} a_n t^n/n!$ be in some DF algebra, with $a_0 = 0$ and $a_1 \neq 0$. Let y be the compositional inverse of x; that is, $t = y(x) = x(y)$. Let $x_1, \ldots, x_k$ generate A; we may assume that $x_1 = t$. Let B be the algebra generated by the series $x_1(y), \ldots, x_k(y)$ and y'. Then B contains $y = x_1(y)$. Moreover, $(x_i(y))' = x_i'(y) \cdot y'$ is in B, because x_i' is in A. Now, $y' = 1/x'(y)$, which implies $y'' = -x''(y)y'(x'(y))^{-2} = -x''(y)y'^3$. Hence, y'' is in B, because x'' is in A, from which we conclude that B is a DF algebra.

The following theorem gives some information on the coefficients of CDF series. We shall use it to give counter-examples.

THEOREM 3. *Let $x = \sum_{n\geq 0} a_n t^n/n!$ be a CDF series. Then:*
(*i*) *The series x is analytic around* 0: *in fact $|a_n| \leq \alpha^n n!$ $(n \geq 1)$ for some constant α.*
(*ii*) *If the a_n's lie in a subfield K of $\mathbb{C}$, then there exists for x a system such as equation* (1) *with coefficients in K.*
(*iii*) (Eisenstein criterion, compare to [16], p. 139) *If the a_n's are rational numbers, then for some integer $d \geq 1$, one has $d^n a_n \in \mathbb{Z}$ $(n \geq 1)$.*

As a consequence of the theorem, we give some counter-examples.

The set of CDF series is not closed for the *Hadamard product* (coefficient by coefficient product). Indeed, the Hadamard square of e^t is $\sum (n!)^{-2} t^n$, which does not satisfy the Eisenstein criterion. Similarly, the set of CDF series is not closed for the product

$$\sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \sum_{n=0}^{\infty} b_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} a_n b_n \frac{t^n}{n!}.$$

Indeed, the square of $(1-t)^{-1}$ for this product is $\sum n! t^n$, which is not analytic around 0.

The series $(e^t - 1)t^{-1} = \sum (n+1)^{-1} t^n/n!$ is not CDF, because it does not satisfy the Eisenstein criterion. This implies, by Theorem 2, that its inverse $t(e^t - 1)^{-1}$, which is the generating function of the *Bernoulli numbers*, is *not* CDF. This may seem strange, at first glance, because $x=(e^t-1)^{-1}$ satisfies $x'=-x-x^2$. The reason why this is so is because x is *not* in $\mathbb{C}[[t]]$ (i.e. (e^t-1) is not invertible in $\mathbb{C}[[t]]$); hence x is not a CDF series.

Proof of Theorem 3. (i) Let x be generated by some tree labelling table with k colors. Let α be the supremum of the absolute values of the coefficients of the table. Thus, the weight of any tree on $[n]$ generated by this table is clearly bounded by α^n. As there are exactly $(n-1)!$ increasing trees on $[n]$, $|a_n|$ is bounded by $\alpha^n(n-1)!\leqslant \alpha^n n!$.

(ii) Let M be the field generated by K and the finite number of coefficients occurring in equation (1). Then M is finitely generated over K; hence there exists an intermediate field L such that L is a pure transcendental extension of K, and M is finite over L. Let $m_1,\ldots,m_h$ be a basis of M over L, with $m_1=1$. Then define kh series x_{ij} $(1\leqslant i\leqslant k,\ 1\leqslant j\leqslant h)$ in $L[[t]]$ by

$$x_i=\sum_{j=1}^{h} m_j x_{ij}.$$

Also define indeterminates t_{ij} $(1\leqslant i\leqslant k,\ 1\leqslant j\leqslant h)$, and hk polynomials P_{ij} in $L[t_{ij}]$ by

$$P_i(m_1t_{11}+\cdots+m_ht_{1h},\ldots,m_1t_{k1}+\cdots+m_ht_{kh})=\sum_{j=1}^{h} m_jP_{ij}(t_{11},\ldots,t_{kh}).$$

Then the series x_{ij} satisfy the following system, with coefficients in L:

$$x'_{ij}=P_{ij}(x_{11},\ldots,x_{kh}),\tag{8}$$

since

$$x'_i=\sum_{j=1}^{h} m_jx'_{ij},$$

implies that

$$P_i(m_1x_{11}+\cdots+m_hx_{1h},\ldots,m_1x_{k1}+\cdots+m_hx_{kh})=\sum_{j=1}^{h} m_jP_{ij}(x_{11},\ldots,x_{kh}).$$

Now, the coefficients of equation (8) are in L, which is of the form $K(\theta_1,\ldots,\theta_l)$ for some θ_i's algebraically independent over K. Let $P(\theta_1,\ldots,\theta_l)$ be a common denominator of the coefficients appearing in (8), and let $\alpha_1,\ldots,\alpha_l\in K$ be such that $P(\alpha_1,\ldots,\alpha_l)\neq 0$. Replacing the θ_i's by the α_i's in equation (8) gives a system over K satisfied by series z_{ij}. As x_1 is in $K[[t]]$, one has $x_1=x_{11}=z_{11}$, and (ii) is proved.

(iii) We may suppose, by (ii), that x is defined by a tree labelling table with coefficients in $\mathbb{Q}$. Let d be a common denominator of these coefficients. Then $d^n\omega\in\mathbb{Z}$, if ω is the weight of any tree on $[n]$ generated by the table. As a_n is the sum of all these weights, we obtain (iii). □

For $x=\sum_{n\geqslant 0} a_nt^n/n!$ a CDF with $a_n\in\mathbb{Z}$, one open question is whether it is possible to find a system with integer coefficients for this series. Note that for a given CDF series there are many systems giving this series. Another open problem is whether it is possible to have unicity modulo some transformation group. This may be related to problems of differential algebra (see [10]).

5. Comparison with Other Families

First of all, we verify that any CDF series x is also differentially algebraic. Indeed, by Lemma 1, the derivatives of x all belong to some finitely generated subalgebra of $\mathbb{C}[[t]]$.

Hence, the infinite set $\{x^{(k)} \mid k \geq 0\}$ is algebraically dependent over $\mathbb{C}$. This directly implies that x is differentially algebraic.

Theorem 3 shows that each rational power series is CDF. We show, using a theorem of Fliess [7] and Christol [4], that this assertion can be extended to algebraic series.

THEOREM 4. *Each algebraic formal power series is CDF.*

PROOF. Let x be algebraic. We may suppose that $x(0) = 0$. By [6, proposition 7], x is *constructively algebraic*: that is, there exist k series $x_1, \ldots, x_k$ with constant term zero and $x_1 = x$, and k polynomials $P_i(T, X_1, \ldots, X_k)$ in $\mathbb{C}[T, X_1, \ldots, X_k]$, each having zero constant term and no monomial of the form X_j, such that

$$x_i = P_i(t, x_1, \ldots, x_k), \qquad i = 1, \ldots, k. \tag{9}$$

Differentiating (9) with respect to t, we obtain

$$x_i'\left[1 - \frac{\partial P_i}{\partial X_i}(t, x_1, \ldots, x_k)\right] - \sum_{j \neq i} x_j' \frac{\partial P_i}{\partial X_j}(t, x_1, \ldots, x_k) = \frac{\partial P_i}{\partial T}(t, x_1, \ldots, x_k), \tag{10}$$

for $i = 1, \ldots, k$. The special properties of the P's and the fact that the x's are without constant term ensure that (10) is a system of linear equations in $x_1', \ldots, x_k'$, the determinant y of which is invertible in $\mathbb{C}[[t]]$. Thus, x_i' is in the algebra A generated by $t, x_1, \ldots, x_k, y^{-1}$. Moreover, y is in the algebra generated by $x_1, \ldots, x_k$; thus $y' \in A$, and it follows that $(y^{-1})' = -y'y^{-2}$ is also in A. This shows that A is a finitely generated algebra which is closed for differentiation, and we conclude that $x = x_1$ is CDF by Lemma 1. □

A series x is called *D-finite* (see [19]) if it satisfies a non-trivial linear differential equation of the form

$$Q_k(t)x^{(k)} + \cdots + Q_1(t)x' + Q_0(t)x = 0 \tag{11}$$

for some polynomials $Q_0, \ldots, Q_k$, with $Q_k \neq 0$.

Equivalently [19, th. 1.5], $x = \sum_{n \geq 0} a_n t^n$ is *P-recursive,* if for some polynomials

$$P_d(n)a_{n+d} + \cdots + P_1(n)a_{n+1} + P_0(n)a_n = 0 \qquad (n \geq 0).$$

We show that D-finite series and CDF series form two incomparable families. Indeed, $(\cos(t))^{-1}$ is CDF, as shown at the beginning of Section 2; however, it is not D-finite, as shown by Carlitz ([15, th. 4]; see also [19, example 2.5 or 4(a)]). Conversely, the series $\sum n!t^n$ is easily shown to be P-recursive, and hence D-finite, but it is not CDF, because it is not analytic in any neighborhood of 0 (see Theorem 2). Another example, which is analytic, is the series

$$(e^t - 1)t^{-1} = \sum (n+1)^{-1} t^n / n!,$$

which is D-finite, but not CDF, as shown in Section 4. Because of the strong relationship between Bernoulli and Genocchi numbers, this leads to the similar question for the exponential generating series of the *Genocchi numbers*

$$\frac{2t}{e^t + 1} = t + \sum_{n=1}^{\infty} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

This series *is* CDF because the series $x = 2(1 + e^t)^{-1}$ is such that $x' = -x + 2^{-1}x^2$. Using equation (5), this equation provides the tree labelling table of Figure 9.

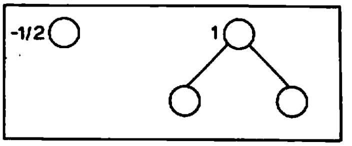

FIGURE 9.

Now, observe that

$$x = 1 + \sum_{n=0}^{\infty} (-1)^{n+1} G_{2n+2} \frac{t^{2n+1}}{(2n+2)(2n+1)!},$$

and that each complete binary tree with $2n+1$ nodes has exactly $n+1$ leaves; and hence, according to the above table, a weight of $(-1/2)^{n+1}$. Thus $(2n+2)^{-1}G_{2n+2}$ is the number of complete binary increasing trees on $[2n+1]$, divided by 2^{n+1}. This, together with Example 2 of Section 3, implies the following relation between Genocchi and Euler numbers: $2^{2n}G_{2n+2} = (n+1)E_{2n+1}$. This is relation (1.6) of [20].

A subclass of CDF series is the set of series x satisfying an equation of the form

$$x^{(k)} = P(x, x', \ldots, x^{(k-1)}) \tag{12}$$

for some polynomial P. This is a subclass because the subalgebra of $\mathbb{C}[\![t]\!]$ generated (compare with Lemma 1) by $x, x', \ldots, x^{(k-1)}$ is closed for differentiation. Clearly, x satisfies an equation of form (12) iff the subalgebra generated by x and all its derivatives is finitely generated. We shall show that this subclass is a proper subset of all CDF series.

The series $\exp(t^2)$ is CDF, since $\exp(t^2)' = 2t\exp(t^2)$. We shall show that $\exp(t^2)$ is not in th subclass of CDF series satisfying an equation of form (12). Recall that $\exp(t^2)$ is not algebraic. If $\exp(t^2)$ were to satisfy an equation of form (12), then the algebra A generated by all the derivatives $(\exp(t^2))^{(n)}$, $n \geq 0$, would be finitely generated. Now, $(\exp(t^2))^{(n)}$ is a linear combination of $\exp(t^2)$, $t \cdot \exp(t^2), \ldots,$ and $t^n \exp(t^2)$, the latter having, moreover, a non-zero coefficient. Hence A is also generated by the series $t^n \exp(t^2)$, $n \geq 0$. Now, A being finitely generated, there exists an N such that A is generated by the series:

$$\exp(t^2), \qquad t \cdot \exp(t^2), \ldots, t^N \exp(t^2).$$

Hence

$$t^{N+1}\exp(t^2) = P(\exp(t^2), t \cdot \exp(t^2), \ldots, t^N \exp(t^2))$$

for some polynomial P. This may be rewritten as

$$\begin{aligned} t^{N+1}\exp(t^2) &= \sum_{i=0}^{k} \sum_{j_0+\cdots+j_N=i} \alpha_{ij} (\exp(t^2))^{j_0} \cdots (t^N \exp(t^2))^{j_N}, \\ &= \sum_{i=0}^{k} (\exp(t^2))^i \sum_{j_0+\cdots+j_N=i} \alpha_{ij} t^{1j_1+\cdots+Nj_N}, \end{aligned}$$

where j means $(j_0, \ldots, j_N)$. But $\exp(t^2)$ is not algebraic, and thus we must have:

$$t^{N+1} = \sum_{j_0+\cdots+j_N=1} \alpha_{ij} t^{1j_1+\cdots+Nj_N}.$$

But this is surely impossible, because t^{N+1} is not a linear combination of $1, t, \ldots, t^N$.

Each CDF series is also differentially algebraic. However, the converse is far from being true. Indeed, there are differentially algebraic series which are not analytic around 0, such as the above considered series $\sum n!t^n$ which is D-finite, and hence also differentially algebraic.

Finally, we mention a decidability result, due to Denef and Lipshitz (which was indicated to us by the latter author). They show that it is decidable if a system of differentially algebraic equations with given initial conditions has a power series solution (see [5, theorem 3.1]). An easy consequence of this is that it is decidable if a given CDF series vanishes: indeed, if $x = x_1$ satisfies equation (1), with the initial conditions $x_i(0) = \alpha_i$, then $x = 0$ iff $\alpha_1 = 0$ and if the system

$$x_2' = P_2(0, x_2, \ldots, x_k),$$
$$\vdots$$
$$x_k' = P_k(0, x_2, \ldots, x_k)$$

with $x_i(0) = \alpha_i$ has a solution. This result shows also that one may decide the equality of two CDF series characterized by systems such as (1) (the 'word problem' for CDF series).

Acknowledgments

Conversations with Pierre Leroux, Gilbert Labelle and Doron Zeilberger were helpful. We also thank Lee Rubel and Leonard Lipshitz for useful correspondence. F.B. was supported by grant NSERC A9041 (Canada), and grant FCAR EQ1608 (Québec).

References

1. F. Bergeron and C. Reutenauer, Combinatorial interpretation of the powers of a linear differential operator, *Ann. Sci. Math. Québec*, 1987, No. 2, 269–278.
2. J. Berstel and C. Reutenauer, *Les séries rationnelles et leurs langages*, Masson, Paris, 1984.
3. L. Carlitz, Recurrences for the Bernoulli and Euler numbers, *J. Reine Angew. Math*, **214/215** (1964), 184–191.
4. G. Christol, Sur une opération analogue à l'opération de Cartier en caractéristique nulle, *C.R. Acad. Sci. Paris*, **271A** (1970), 1–3.
5. J. Denef and L. Lipschitz, *Power series solutions of algebraic differential equations, Math. Ann.*, **267** (1984), 213–238.
6. M. Fliess, Sur divers produits de séries formelles, *Bull. Soc. Math. Fr.*, **102** (1974), 181–191.
7. M. Fliess, Fonctionnelles causales non linéaires et indéterminées non commutatives, *Bull. Soc. Math. Fr.*, **109** (1981), 3–40.
8. D. Foata, *La série génératrice exponentielle dans les problèmes d'énumération*, Presses Univ. Montréal, 1974.
9. W. Gröbner, *Die Lie Reihen und ihre Anwendungen*, Deutscher Verlag der Wiss. Berlin, 1967.
10. E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Academic Press, New York, 1973.
11. G. Labelle, Eclosions combinatories appliquées à l'inversion multidimensionnelle des séries formelles, *J. Combin. Theory Ser., A*, **39** (1985), 52–82.
12. P. Leroux and G. Viennot, Combinatorial resolution of differential equations, I: ordinary differential equations, in: *Proc. Colloque de Combinatoire*, U.Q.A.M., Montréal, 1985; Lect. Notes in Math. No. 1234, Springer-Verlag, Berlin, pp. 210–245.
13. P. Leroux and G. Viennot, Résolution combinatoire d'équations différentielles II: calcul intégral combinatoire, *Ann. Sci. Math. Québec*, **12** (1988) 233–253.
14. P. Leroux and G. Viennot, Combinatorial resolution of differential equations IV: separation of variables, *Discr. Math*, **72** (1988) 237–250.
15. E. Maillet, Sur les séries divergentes et les équations différentielles, *Ann. Sci. Ecole Normale Sup. Sér. 3*, **20** (1903), 487–518.

16. G. Polyá and G. Szegö, *Aufgaben und Lehrsätze ans der Analysis II*, Springer-Verlag, Berlin, 4. Auflage, 1971.
17. J. Popken, *Über arithmetische Eigenschaften analyscher Funktionen*, North-Holland, Amsterdam, 1935.
18. L. Rubel, An elimination theorem for systems of algebraic differential equations, *Houston J. Math.*, **8** (1982), 289–295.
19. R. P. Stanley, Differentiably finite power series, *Europ. J. Combin.*, **1** (1980), 175–188.
20. G. Viennot, Interprétation combinatoire des nombres d'Euler et de Genocchi, Séminaire de Théorie des Nombres 1980–1981, Paris.

Received 19 January 1987; accepted in revised form 30 April 1990

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