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Cyclic Derivation of Noncommutative Algebraic Power Series

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We show that the cyclic derivative of any algebraic formal power series in noncommuting variables is again algebraic.

INTRODUCTION

Rota *et al.* [8] have introduced two operators C and D of the algebra of noncommutative formal power series; they are defined in the following way: the image Cw of a word w is

$$Cw = \sum_{\substack{uv=w \\ v \neq 1}} vu$$

(and C is extended by linearity and continuity) and the *cyclic derivation* D is obtained by composing C with the map T such that

$$T(xw) = w,$$

$$T(x) = 0$$

if w does not begin with x where x is a fixed letter. In the case of one variable, D is the usual derivation. The above cited authors show that the image under C or D of any rational noncommutative power series is again rational. The purpose of the present note is to prove that the image of any *algebraic* power series is still algebraic (which is well known for one variable): the concept of algebraicity for noncommutative formal power series that we shall use here was introduced by Chomsky and Schützenberger [3]; it extends on one hand the usual concept in one variable and, on the other, the concept of context-free languages. These languages are closed with

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respect to C (and the same is true for rational (or regular) languages, as proved by Schützenberger (unpublished result, cited in [5, II.23])).

1. Preliminaries

Let X be an alphabet, X^* the *free monoid* generated by X. The neutral element of X^* is denoted by 1. The elements of X are *words* and 1 is the *empty word*. The length of a word w is denoted by $|w|$.

Let K be a field; the *algebra of noncommutative polynomials* $K\langle X\rangle$ is the set of all linear combinations

$$P = \sum_{w \in X^*} (P, w) w,$$

where (P, w) denotes the coefficient of the word w; $I = K^+\langle X\rangle$ denotes the ideal of the polynomials P such that $(P, 1) = 0$ (i.e., without constant term). The powers

$$I^n, \qquad n \geqslant 0,$$

of I define a fundamental set of neighbourhoods of 0 for a metrizable topology, see [8]. The completion of $K\langle X\rangle$ is the *algebra of noncommutative formal power series* $K\langle\langle X\rangle\rangle$, which can be identified with the (finite or infinite) sums

$$S = \sum_{w \in X^*} (S, w) w.$$

A language L is a subset of X^*; it can be identified with its *characteristic* series

$$\underline{L} = \sum_{w \in L} w.$$

The support of a formal power series S is the language

$$\operatorname{supp}(S) = \{w \in X^* \mid (S, w) \neq 0\}.$$

A formal power series is *rational* if it can be obtained from polynomials by a finite number of the following operations: sum, product, inversion. By a theorem of Schützenberger (which extends a well-known theorem of Kleene) a series is rational if and only if it is *recognizable*, that is, if there exist $n \geqslant 1$, an algebra homomorphism

$$\mu : K\langle X\rangle \rightarrow \mathscr{M}_n(K)$$

and matrices $\lambda \in \mathscr{M}_{1,n}(K)$, $\gamma \in \mathscr{M}_{n,1}(K)$ such that for each word w

$$(S, w) = \lambda\mu w\gamma$$

(for a proof of this theorem, see [4] or [9]).

Let us now define algebraic series. Let

$$\Xi = \{\xi_1,..., \xi_n\}$$

be a new alphabet and

$$P_1(\xi_1,..., \xi_n),..., P_n(\xi_1,..., \xi_n) \in K\langle X \cup \Xi\rangle.$$

We say that the system

$$\begin{aligned} \xi_1 &= P_1(\xi_1,..., \xi_n) \\ &\vdots \\ \xi_n &= P_n(\xi_1,..., \xi_n) \end{aligned} \qquad (1.1)$$

is an *algebraic system of equations*. We now introduce two conditions on the polynomials P_i, each of which is sufficient to imply the existence and the uniqueness of an n-tuple of formal power series in $K\langle\langle X\rangle\rangle$ which is a solution of this system.

Call the system *proper* if for each i

$$\operatorname{supp}(P_i) \cap \{1, \xi_1,..., \xi_n\} = \varnothing.$$

Call the system *strict* if for each i

$$\operatorname{supp}(P_i) \cap \Xi^* = \varnothing.$$

If the system is strict, it can be shown that there exists one and only one n-tuple $(S_1,..., S_n) \in K\langle\langle X\rangle\rangle^n$ which is *solution* of this system, that is, such that

$$\begin{aligned} S_1 &= P_1(S_1,..., S_n) \\ &\vdots \\ S_n &= P_n(S_n,..., S_n), \end{aligned} \qquad (1.2)$$

where $P_i(S_1,..., S_n)$ denotes the series obtained by replacing each ξ_j in P_i by S_j. Similarly, if the system is proper, there exists one and only one n-tuple $(S_1,..., S_n) \in K^+\langle\langle X\rangle\rangle^n$ (where $K^+\langle\langle X\rangle\rangle$ is the set of all series without constant term) such that one has (1.2); this n-tuple will be called the *proper solution* of the system.

One shows that the following conditions are equivalent for a formal power series S:

(i) S is a component of the solution of a strict system,

(ii) S is the sum of a constant and component of the proper solution of a proper system

(See [9, Chapter IV, Theorem 1.1, Definition p. 120 and Theorem 2.3].) In this case, we say that S is *algebraic*.

Let $C: K\langle\langle X\rangle\rangle \to K\langle\langle X\rangle\rangle$ be the K-linear and continuous mapping defined for each word $w = x_1 \cdots x_n$ $(x_i \in X)$ by

$$Cw = x_1 \cdots x_n + x_2 \cdots x_n x_1 + \cdots + x_n x_1 \cdots x_{n-1} = \sum_{\substack{w = uv \\ v \neq 1}} vu; \qquad C1 = 1.$$

Rota *et al.* [8] show that if S is rational, then CS is again rational. We want to deduce the following result.

THEOREM. *If S is algebraic, then CS is algebraic.*

The above-cited authors call *cyclic derivation* with respect to a given letter x_0, the operator D of $K\langle\langle X\rangle\rangle$ defined by

$$DS = x_0^{-1}(CS),$$

where, for each series T, $x_0^{-1}T$ is defined by

$$x_0^{-1}T = \sum_{w \in X^*} (T, x_0 w)\, w.$$

Now it is a classical result that if T is algebraic then $x_0^{-1}T$ is algebraic too (e.g., it is a consequence of Theorem 2.3. Chapter IV in [9]). Hence, the theorem above implies the

COROLLARY. *If S is algebraic, DS is algebraic.*

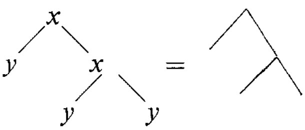
EXAMPLE. Consider the series defined by

$$S = xSS + y.$$

We have

$$S = x + xyy + xxyyy + xyxyy + xxxyyyy + \cdots .$$

S is just the characteristic series of the well-known Luckasiewicz language L (see [2, II.4]). A word in L can be pictured as a binary tree; for instance,

$xyxyy =$ 

Let $|w|_x$ denote the degree in x of w. Then for each w such that $|w|_x - |w|_y = -1$, there exists one and only one $u \in L$ conjugate to w (i.e., $u = ab$, $w = ba$), see [2, Proposition II.4.3]. Hence

$$CS = \sum_{|w|_x - |w|_y = -1} w$$

and, letting D denote the cyclic derivation with respect to y,

$$DS = \sum_{|w|_x - |w|_y = 0} w,$$

which is algebraic (ibid.)

Remarks. 1. The operator $x^{-1}T$ allows one to characterize rationality in a nice way: indeed a series S is recognizable if and only if the smallest subspace of $K\langle\langle X\rangle\rangle$ containing S and closed with respect to the operators x^{-1} (for all letters x) is finite dimensional. This dimension is the *rank* of S, which can also be defined by a Hankel matrix (see [4]).

2. Through the notion of recognizability, one can give another proof of the result of Rota *et al.*: indeed if

$$(S, w) = \lambda\mu w\gamma \qquad \text{(see above)}$$

then

$$(CS, w) = \sum_{\substack{w = uv \\ v \neq 1}} (S, vu) = \sum_{\substack{uv = w \\ v \neq 1}} \sum_{1 \leqslant i \leqslant n} (\mu u\gamma)_i (\lambda\mu v)_i$$
$$= \sum_{i,u,v} (S_i, u)(T_i, v) = \sum_i (S_i T_i, w),$$

where S_i and T_i are the recognizable (hence rational) series defined by

$$(S_i, w) = \lambda_i \mu w\gamma, \qquad (T_i, w) = \lambda\mu w\gamma_i,$$

and where λ_i (resp. γ_i) are the matrices of the canonical basis of $\mathscr{M}_{1,n}(K)$ (resp. $\mathscr{M}_{n,1}(K)$). Hence $CS = \sum_i S_i T_i$ is rational.

2. Proof of the Theorem

The *Hadamard product* of two series S and T is the series

$$S \odot T = \sum_{w} (S, w)(T, w)\, w.$$

Let Y and $\bar{Y}$ be two alphabets and

$$\begin{aligned} y &\mapsto \bar{y} \\ Y &\to \bar{Y} \end{aligned}$$

be a bijection between them. We call *Dyck language* the language Δ on the alphabet $Z = Y \cup \bar{Y}$ defined by

$$\Delta = \varphi^{-1}(1),$$

where φ is the canonical morphism

$$\begin{aligned} &Z^* \to Y^{(*)}, \\ &y \mapsto y, \\ &\bar{y} \mapsto y^{-1} \end{aligned}$$

and $Y^{(*)}$ the free group generated by Y. We still denote by Δ the characteristic series of the Dyck language: Δ is an algebraic series (see [2, II.3)]. The theorem of Chomsky–Schützenberger [3] asserts that for each algebraic series $S \in K\langle\langle X\rangle\rangle$ there exists an alphabet $Z = Y \cup \bar{Y}$, a rational series $R \in K\langle\langle Z\rangle\rangle$ and an alphabetical morphism $\psi\colon Z^* \to X^*$ (alphabetical means that the image under ψ of each letter is either a letter or the empty word) such that the family of series

$$((R \odot \Delta, w)\, \psi(w))_{w \in Z^*}$$

is summable in $K\langle\langle X\rangle\rangle$ and that its sum is S. We then write

$$S = \psi(R \odot \Delta).$$

(For a proof of this result see [9, IV.4].)

There exists a more sophisticated version of this theorem: it says that there exists a *rational language* K (see [4] or [2]) and an integer k such that $\operatorname{supp}(R) \subset K$ and that

$$\forall\, aub \in K, \qquad |u| \geqslant k \Rightarrow |\psi(u)| \geqslant 1. \tag{2.1}$$

In other words, each factor of length at least k of any word in K contains at least one letter $z \in Z$ such that $\psi(z) \in X$. This improvement of the Chomsky–Schützenberger theorem is proved for languages in [1] and [2] (Ex. 3.8 of Chapter II), but its extension to formal power series is straightforward.

Now, let $S \in K\langle\langle X\rangle\rangle$ an algebraic series and Y, ψ, R, K and k as above. Let K' be the language defined by

$$K' = CK = \{vu \mid uv \in K\}.$$

Condition (2.1) implies that a similar condition is true for K' (with $2k$ in place of k); indeed each factor of length at least $2k$ of a work in K' contains a factor of length k of some word in K.

Let Z_1 be the set of all letters $z \in Z$ such that $\psi(z) = 1$ and $Z_2 = Z \backslash Z_1$. We denote by T the series

$$T = C(\Delta \odot R) \odot \underline{Z^* Z_2}$$

(recall that $\underline{Z^*Z_2}$ is the characteristic series of the language Z^*Z_2). Then $\text{supp}(R) \subset K$ implies that $\text{supp}(T) \subset K'$, hence the family of series $((T, w)\, \psi(w))_{w \in Z^*}$ is summable in $K\langle\langle X\rangle\rangle$; we denote by $\psi(T)$ its sum and show now that

$$\psi(T) = CS. \tag{2.2}$$

Indeed, let $w = u_0 z_1 u_1 \cdots z_n u_n$, $u_i \in Z_1^*$, $z_i \in Z_2$ with $\psi(z_i) = x_i \in X$. Then

$$C\psi(w) = x_1 \cdots x_n + x_2 \cdots x_n x_1 + \cdots + x_n x_1 \cdots x_{n-1}$$

and

$$\begin{aligned} \psi(Cw \odot \underline{Z^*Z_2}) &= (u_n u_0 z_1 u_1 \cdots z_n + u_{n-1} z_n u_0 z_1 u_1 \cdots z_{n-1} + \cdots \\ &\quad + u_1 \cdots z_n u_n u_0 z_1) \\ &= x_1 \cdots x_n + x_n x_1 \cdots x_{n-1} + \cdots + x_2 \cdots x_n x_1 \end{aligned}$$

hence $C\psi(w) = (Cw \odot \underline{Z^*Z_2})$ and (2.2) follows by linearity and continuity. We now show that $\psi(T)$ is algebraic. Let

$$\begin{aligned} \tau \colon Z^* &\to K\langle\langle X\rangle\rangle, \\ w &\mapsto \psi(w) \quad \text{if} \quad w \in K', \\ w &\mapsto 0 \quad \text{if} \quad w \notin K'. \end{aligned}$$

Because of the condition on K', the mapping τ extends by linearity and continuity to a mapping

$$\tau \colon K\langle\langle X\rangle\rangle \to K\langle\langle X\rangle\rangle$$

which is a *rational regulated transduction* (see [7] or [9, Chapter 3, Section 1]). Hence (ibid.) the image under τ of any algebraic series is algebraic. Now, because $\mathrm{supp}(T) \subset K'$ we have $\tau(T) = \psi(T)$. Furthermore, because $uv \in \Delta \Leftrightarrow vu \in \Delta$ we have

$$C(\Delta \odot R) = \Delta \odot CR$$

and: CR is rational [8], $\underline{Z^*Z_2}$ is rational because

$$\underline{Z^*Z_2} = (1 - \underline{Z})^{-1}\, \underline{Z}_2,$$

$T = \Delta \odot CR \odot \underline{Z^*Z_2}$ is algebraic by a theorem of Schützenberger [10] (the Hadamard product of an algebraic (resp. rational) series by a rational series is again algebraic (resp. rational)).

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