Theoretical Computer Science 32 (1984) 339–340
North-Holland

NOTE

ON FORMAL POWER SERIES DEFINED BY INFINITE LINEAR SYSTEMS

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Communicated by D. Perrin
Received February 1984

Abstract. We show that each formal power series in noncommuting variables may be obtained by an infinite linear system as those considered by Kuich and Urbanek (1983).

In a recent paper, Kuich and Urbanek [2] have introduced infinite linear systems of noncommutative formal power series. The aim of the present note is to show that each formal power series may be obtained by such a system. As a consequence, the closure properties given in [2] become straightforward.

We follow the notations and definitions of [2] and [3]. Let A be a semi-ring and Σ be an alphabet. An infinite matrix is called *row-finite* (resp. *column-finite*) if in each of its rows (resp., columns), there are only a finite number of nonzero coefficients. An infinite linear system is defined to be of the form

$$Y = P + MY, \tag{1}$$

where Y is an $\mathbb{N}$ by 1 column vector of variables, P an $\mathbb{N}$ by 1 column-finite column vector with coefficients in $A\langle\langle\Sigma^*\rangle\rangle$, and M an $\mathbb{N}$ by $\mathbb{N}$ row- and column-finite matrix with coefficients in $A\langle\langle\Sigma^*\rangle\rangle$. The system is called *cycle-free* if, setting $M = M_0 + M_1$ with $M_0 = (M, \lambda)\lambda$ (where λ denotes the empty word), the matrix M_0 is nilpotent and M_1 is quasi-regular. By [2, Theorem 1] each cycle-free infinite linear system has a unique solution.

We show that each formal power series may be obtained as the first component of such a system. For this, let $S \in \langle\langle\Sigma^*\rangle\rangle$ any formal power series. We adopt, as in [1], the language of A-Σ-automata, rather than systems. Let $\boldsymbol{\Sigma}$ be a disjoint copy of Σ: the natural bijection

$$\Sigma^* \to \boldsymbol{\Sigma}^*$$

will be denoted by

$$w \to \boldsymbol{w}.$$

We define an automaton with $Q = \Sigma^* \cup \boldsymbol{\Sigma}^*$ as set of states. The initial state is λ, with label 1; there are two final states, λ with label (S, λ) and $\boldsymbol{\lambda}$ with label 1. For any word w in Σ^* and any letter σ in Σ, there are two edges with label σ and multiplicity 1,

$$w \to w\sigma \quad \text{and} \quad \boldsymbol{\sigma w} \to \boldsymbol{w}.$$

For any words u and v in Σ^* such that

$$|u| = |v| \quad \text{or} \quad |u| = |v| + 1,$$

and for any letter σ in Σ such that $(S, u\sigma v) \neq 0$ there is an edge labelled σ with multiplicity $(S, u\sigma v)$,

$$u \to \boldsymbol{v}.$$

Note that this automaton is locally-finite, that is, for each state, there are only a finite number of edges going in it (respectively going out of it). Furthermore, there are only a finite number (namely, 2) of final states. This implies that the associated system satisfies the row- and column-finiteness conditions. This system is cycle-free because there are no λ-transitions in the automaton. Now, it is easy to verify that for each nonempty word w in Σ^*, there is only one succesful path with label w, and it has multiplicity (S, w). Furthermore, the only successful path with label λ has multiplicity (S, λ). This shows that the series recognized by this automaton (hence the series which is the first component of the solution of the associated system) is S.

References

[1] S. Eilenberg, *Automata, Languages and Machines, Vol A* (Academic Press, New York, 1974).

[2] W. Kuich and F.J. Urbanek, Infinite linear systems and one counter languages, *Theoret. Comput. Sci.* **22** (1983) 95–126.

[3] A. Salomaa and M. Soittola, *Automata-theoretic Aspects of Formal Power Series* (Springer, Berlin, 1978).