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Rationality of the Möbius function of subword order

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Abstract

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We prove the rationality of various noncommutative formal power series, whose coefficients are determined by the Möbius function or zeta function of the subword partial order of noncommutative monomials. We also give explicit expressions for the corresponding commutative generating functions.

0. Introduction

Noncommutative rational formal series naturally arise from automata theory and language theory (see [1, 5]). Less well-known is the rationality of series arising from combinatorics. We give here examples of such series. We show that the zeta function and the Möbius function of subword order, and their powers, when viewed as formal series of words, are rational.

We give also explicit forms for the corresponding commutative generating series. The techniques used here allow us to give another proof of the combinatorial interpretation, given in [2], of the Möbius function.

1. Möbius function of subword order

Let A^* denote the free monoid over an alphabet A. The elements of A^* are called *words*. The *length* $|\alpha|$ of a word α is the number of letters in α. The empty word, of

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length 0, is denoted by 1. A word β is a *subword* of a word α if $\alpha = a_1 \dots a_n$ ($a_i \in A$, $n \geqslant 0$), and if there exists an *embedding* of β in α, i.e. a sequence

$$(1.1) \quad 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n \quad (k \geqslant 0)$$

such that $\beta = a_{i_1} \dots a_{i_k}$. We write $\beta \leqslant \alpha$ if β is a subword of α. This is a partial ordering of A^*. The *incidence algebra* $\mathscr{I}$ of this partial ordering is the set of all formal infinite linear combinations over $\mathbb{Z}$

$$\sum_{\beta \leqslant \alpha} c(\beta, \alpha) \beta \otimes \alpha,$$

where the coefficient $c(\beta, \alpha)$ of $\beta \otimes \alpha$ is in $\mathbb{Z}$. The product in $\mathscr{I}$ is defined by

$$(1.2) \quad \left(\sum_{\beta \leqslant \alpha} c(\beta, \alpha) \beta \otimes \alpha \right) \left(\sum_{\beta \leqslant \alpha} d(\beta, \alpha) \beta \otimes \alpha \right) = \sum_{\beta \leqslant \alpha} \left(\sum_{\beta \leqslant \gamma \leqslant \alpha} d(\beta, \gamma) c(\gamma, \alpha) \right) \beta \otimes \alpha.$$

The *zeta function* of $\leqslant$ is the element of $\mathscr{I}$ whose coefficients are all equal to one, i.e.

$$\zeta = \sum_{\beta \leqslant \alpha} \beta \otimes \alpha.$$

The *Möbius function* is defined to be the inverse of ζ in $\mathscr{I}$. Thus,

$$\mu = \sum_{\beta \leqslant \alpha} \mu(\beta, \alpha) \beta \otimes \alpha,$$

where the coefficients $\mu(\beta, \alpha)$ are, in view of (1.2), defined by

$$(1.3) \quad \sum_{\beta \leqslant \gamma \leqslant \alpha} \mu(\gamma, \alpha) = 0$$

for any words β, α, such that $\beta < \alpha$, and $\mu(\alpha, \alpha) = 1$. See [2] for references to the literature on Möbius functions.

The coefficients $\mu(\gamma, \alpha)$ have a combinatorial interpretation. For a word α as above, call *repetition set* of α the set

$$\mathscr{R}(\alpha) = \{ i \mid 2 \leqslant i \leqslant n, \ a_i = a_{i-1} \}.$$

For example, $\mathscr{R}(aaababb) = \{2, 3, 7\}$. A *normal embedding* of the word β in α is a sequence as (1.1) which contains the repetition set. The coefficient $\binom{\alpha}{\beta}_{\mathrm{n}}$ is defined to be the number of normal embedding of β in α. E.g. $\binom{aaababb}{aaab}_{\mathrm{n}} = 2$.

Remarks. (1) We follow Eilenberg's notation in ([3, VIII.10]): he defines the *binomial coefficient* $\binom{\alpha}{\beta}$ as the number of embeddings of β in α.

(2) The definition of $\binom{\alpha}{\beta}_{\mathrm{n}}$ is asymmetric only for the simplicity of the presentation. This number does not change if one replaces $a_i = a_{i-1}$ by $a_i = a_{i+1}$ in the definition of $\mathscr{R}(\alpha)$.

The following result characterizes the Möbius function of subword order.

Theorem 1.1 (Björner [2]). *The following relation holds for any words* α, β:

$$\mu(\beta, \alpha) = (-1)^{|\alpha| + |\beta|} \binom{\alpha}{\beta}_{\mathrm{n}}$$

We give an algebraic proof of this result in the next section, different from the proof in [2], which used lexicographic shellability.

2. Noncommutative formal series

Let $\mathbb{Z}\langle\langle A\rangle\rangle$ denote the algebra over $\mathbb{Z}$ of noncommutative formal series in the variables ("letters") in A. A series S is denoted by $S=\sum_{w\in A^*}(S,w)w$; (S,w) is the coefficient of the word w. We consider the algebra $\mathscr{A}$ of continuous linear endomorphisms of $\mathbb{Z}\langle\langle A\rangle\rangle$, where the continuity is meant with respect to the usual A-adic topology. Continuity means, in other words, that such a mapping f is completely described by a family of series $(f(\beta))_{\beta\in A^*}$ which satisfies the local finiteness condition: for any word α, there are only a finite number of words β such that the coefficient $(f(\beta),\alpha)$ is nonzero. In this case, the image $f(S)$ of a series S under f is well-defined by the formula

$$f(S)=\sum_{\beta}(S,\beta)f(\beta).$$

A special case of continuous endomorphism is the case: $f(\beta)=\beta+$ linear combination of words longer than β. Such an endomorphism always has an inverse.

Lemma 2.1. *There is a natural embedding of the incidence algebra $\mathscr{I}$ of subword order into $\mathscr{A}$. This mapping is given by*

$$\sum_{\beta\leqslant\alpha} c(\beta,\alpha)\beta\otimes\alpha\mapsto f,$$

where f is defined by

$$f(\beta)=\sum_{\beta\leqslant\alpha} c(\beta,\alpha)\alpha.$$

The proof is immediate, using the definition (1.2) of the product in $\mathscr{I}$.
Denote by z the mapping in $\mathscr{A}$ defined by

$$z(\beta)=\sum_{\beta\leqslant\alpha}\alpha. \tag{2.1}$$

In other words, z is the image under the embedding of Lemma 2.1 of the zeta function.
Denote by m the mapping defined by

$$m(\beta)=\sum_{\alpha}\binom{\alpha}{\beta}_n\alpha. \tag{2.2}$$

Let L be the language (subset of A^*) of words having empty repetition set. For any letter a, let $L_a=\{\alpha\in L\mid\alpha\notin aA^*\}$, that is, the repetition-free words which do not begin with a. We identify a language with its characteristic series in $\mathbb{Z}\langle\langle A\rangle\rangle$.

Define further two continuous algebra endomorphisms φ and ψ of $\mathbb{Z}\langle\langle A\rangle\rangle$ by

$$\varphi(a)=a(A-a)^*,\qquad \psi(a)=aL_a,$$

where, as usual, we use the notation

$$S^* = \sum_{n \geqslant 0} S^n = (1-S)^{-1}$$

for a series S without constant term.

Lemma 2.2. *The following relations hold for any word β:*

$$z(\beta) = A^* \varphi(\beta), \qquad m(\beta) = L\psi(\beta).$$

Proof. Let $\beta = a_1 \ldots a_n$, $a_i \in A$. Then $A^*\varphi(\beta) = A^* a_1 (A - a_1)^* \ldots a_n (A - a_n)^*$. By inspection, one sees that this series has only coefficients 0 or 1, and that a word α appears in this series if and only if β is a subword of α (use the final embedding of β in α). Hence, it is equal to $z(\beta)$. Moreover, $L\psi(\beta) = La_1 L_{a_1} \ldots a_n L_{a_n}$. A similar inspection shows that the coefficient of α in this series is $\binom{\alpha}{\beta}_{\mathrm{n}}$. □

Let i be the involution of $\mathbb{Z}\langle\langle A \rangle\rangle$ (an algebra endomorphism), which sends each word β onto $(-1)^{|\beta|}\beta$. Let $\bar{m} = i \circ m \circ i$. Then, by (2.2):

$$\bar{m}(\beta) = \sum_{\alpha} (-1)^{|\alpha| + |\beta|} \binom{\alpha}{\beta}_{\mathrm{n}} \alpha$$

Thus, by Lemma 2.1 in order to prove Theorem 1.1, we have only to show that $\bar{m}$ is the inverse of z, under composition in $\mathscr{A}$.

Proof of Theorem 1.1. In view of the above remarks, it is enough to show that

$$\bar{m} \circ z = \mathrm{id}.$$

Denoting $\bar{\psi} = i \circ \psi \circ i$ and $\bar{L} = i(L)$, $\bar{L}_a = i(L_a)$, we have

$$\bar{m}(\beta) = i(L\psi \circ i(\beta))$$
$$= \bar{L}\bar{\psi}(\beta).$$

So we have to show that for any word β, one has

$$\bar{L}\bar{\psi}(A^*\varphi(\beta)) = \beta.$$

This is equivalent, because $\bar{\psi}$ is an algebra endomorphism, to

$$\bar{L}\bar{\psi}(A)^* \bar{\psi} \circ \varphi(\beta) = \beta.$$

Hence, it is enough to show that $\bar{\psi}$ is the inverse of φ and that $\bar{L}\bar{\psi}(A)^* = 1$. Now, we clearly have

$$L = 1 + \sum_{b \in A} bL_b;$$

hence, for any letter a

$$L_a = 1 + \sum_{b \neq a} bL_b,$$

which implies

$$\bar{L}_a = 1 - \sum_{b \neq a} b\bar{L}_b.$$

Thus,

$$\bar{L}_a \left(\sum_{b \neq a} b\bar{L}_b \right)^* = 1.$$

Note that

$$\begin{aligned} \bar{\psi}(b) &= i \circ \psi \circ i(b) \\ &= -i \circ \psi(b) \\ &= -i(bL_b) \\ &= b\bar{L}_b. \end{aligned}$$

Thus, we have for each letter $a \in A$,

$$\begin{aligned} a &= a\bar{L}_a \left(\sum_{b \neq a} b\bar{L}_b \right)^* \\ &= \bar{\psi}(a) \left(\sum_{b \neq a} \bar{\psi}(b) \right)^* \\ &= \bar{\psi}(a)\bar{\psi}(A-a)^* \\ &= \bar{\psi}(a(A-a)^*) \\ &= \bar{\psi} \circ \varphi(a). \end{aligned}$$

As $\bar{\psi}, \varphi$ are continuous algebra endomorphisms, we deduce that $\bar{\psi} \circ \varphi = \mathrm{id}$.

Finally, we have

$$\begin{aligned} L = 1 + \sum_{a \in A} aL_a \;&\Rightarrow\; \bar{L} = 1 - \sum a\bar{L}_a \\ &\Rightarrow\; \bar{L} = 1 - \bar{\psi}(A) \\ &\Rightarrow\; \bar{L}\bar{\psi}(A)^* = 1, \end{aligned}$$

which completes the proof. □

3. Rationality

Recall that a formal series in $\mathbb{Z}\langle\langle A \rangle\rangle$ is called *rational* if it may be obtained from polynomials (series with finite support) by applying algebraic operations and the star operation * (or equivalently, inversion). Rational series on the monoid $A^* \times A^*$ are defined similarly; see [1, 5] for details. We will use the notation $\alpha \otimes \beta$ for elements of the direct product $A^* \times A^*$. It was asked by M.P. Schützenberger in 1984 (private communication to the first author) whether the series $\sum \mu(\beta, \alpha)\alpha$ are rational for all $\beta \in A^*$. This will now be verified.

Theorem 3.1. *The series*

$$\sum_{\beta \leqslant \alpha} \beta \otimes \alpha, \qquad \sum_{\beta, \alpha} \binom{\alpha}{\beta}_{\mathrm{n}} \beta \otimes \alpha$$

are rational. Similarly, for any word β, the series

$$\sum_{\beta \leqslant \alpha} \alpha, \qquad \sum_{\alpha} \binom{\alpha}{\beta}_{\mathrm{n}} \alpha$$

are rational.

This is a consequence of Lemmas 2.2, 3.2 and 3.3.

Lemma 3.2. *Let S be a rational series in $\mathbb{Z}\langle\langle A \rangle\rangle$ and θ a continuous algebra endomorphism of $\mathbb{Z}\langle\langle A \rangle\rangle$ such that for any letter a, $\theta(a)$ is a rational series. Then the series $S\theta(\beta)$ is rational for any word β. Moreover, the series $\sum_\beta \beta \otimes S\theta(\beta)$ is rational.*

Proof. This is immediate for the first series, θ preserving multiplication. For the second, we have

$$\sum_\beta \beta \otimes S\theta(\beta) = (1 \otimes S) \sum_\beta \beta \otimes \theta(\beta)$$
$$= (1 \otimes S) \left(\sum_{a \in A} a \otimes \theta(a) \right)^*$$

which is rational. □

Lemma 3.3. *The series L and L_a are rational for any letter a.*

Proof. These series satisfy the linear equations

$$L = 1 + \sum_{b \in A} bL_b \quad \text{and} \quad L_a = 1 + \sum_{b \neq a} bL_b;$$

hence, they are rational by a general theorem (see [3, Prop. VII.6.3]). □

Proof of Theorem 3.1. By Lemma 2.2 we have

$$\sum_{\beta \leqslant \alpha} \alpha = A^* \varphi(\beta) \quad \text{and} \quad \sum_{\alpha} \binom{\alpha}{\beta}_{\mathrm{n}} \alpha = L\psi(\beta).$$

Hence, everything follows via Lemma 3.2, since all basic series involved are rational by Lemma 3.3. □

Remarks. (1) The fact that the third series of the theorem is rational may also be deduced from the fact that the language $\{\alpha \mid \beta \text{ subword of } \alpha\}$ is recognizable by a finite automaton. Automata for the series in Lemma 3.3 are also easily constructed, showing their rationality.

(2) An automaton for the series $\sum \binom{\alpha}{\beta}_n \beta \otimes \alpha$ can also fairly easily be constructed, see Fig. 1.

(3) It is possible to obtain rational expressions for the series of Theorem 3.1, when an explicit alphabet is given. It suffices to apply the usual algorithms for solving linear equations (see the proof of Lemma 3.3), and to follow the proof of Lemma 3.2.

(4) The fact that the series $\sum \binom{\alpha}{\beta} \beta \otimes \alpha$ and $\sum_\alpha \binom{\alpha}{\beta} \alpha$, with the usual binomial coefficients, are rational, is folklore. The first one is equal to

$$\left(\sum_{a \in A} (1+a) \otimes a \right)^*,$$

while the second is

$$A^* a_1 A^* \ldots a_n A^*$$

for $\beta = a_1 \ldots a_n$, $a_i \in A$.

(5) It is also well-known that $\binom{\alpha}{\beta}$ may be defined by the Magnus transformation, which is the continuous algebra endomorphism M sending each letter a onto $1+a$. Hence, for $\alpha = a_1 \ldots a_n$ $(a_i \in A)$

$$M(a_1 \ldots a_n) = (1+a_1) \ldots (1+a_n) = \sum_{\beta \in A^*} \binom{\alpha}{\beta} \beta.$$

A similar expansion holds for

$$m'(\alpha) = \sum_{\beta \in A^*} \binom{\alpha}{\beta}_n \beta.$$

Indeed, it is easily verified that for any letters a, b,

$$m'(\alpha ab) = \begin{cases} m'(\alpha a) b & \text{if } b = a, \\ m'(\alpha a)(1+b) & \text{if } b \neq a. \end{cases}$$

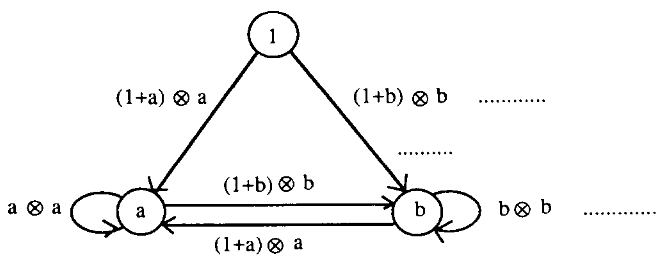

Fig. 1.

For positive integers d, denote by ζ^d the dth power of the zeta function in the incidence algebra of subword order, and similarly for μ^d, the dth power of the Möbius function. Note that $\zeta^d(\beta,\alpha)$ equals the number of chains $\beta\leqslant\beta_1\leqslant\cdots\leqslant\beta_{d-1}\leqslant\alpha$.

Theorem 3.4. *For any $d\geqslant 1$, the series*

$$\sum_{\beta,\alpha}\zeta^d(\beta,\alpha)\beta\otimes\alpha, \qquad \sum_{\beta,\alpha}\mu^d(\beta,\alpha)\beta\otimes\alpha$$

are rational. Similarly, for any word β, the series

$$\sum_{\alpha}\zeta^d(\beta,\alpha)\alpha, \qquad \sum_{\alpha}\mu^d(\beta,\alpha)\alpha$$

are rational.

It is possible to deduce this result from Theorem 3.1, using the methods of Jacob [4, Theorem 1, p. 232]; (see also [5, III.1]). However, we give an independent proof.

Proof. The result will follow, as in the proof of Theorem 3.1, from Lemma 3.2, once it is shown that the mappings z^d, m^d (which correspond to ζ^d and μ^d in the embedding of Lemma 2.1), are continuous linear mappings of the form

$$\sigma:\beta\mapsto S\theta(\beta)$$

for some continuous algebra endomorphism θ of $\mathbb{Z}\langle\langle A\rangle\rangle$ such that $\theta(a)$ is rational for any letter a, and some rational series S in $\mathbb{Z}\langle\langle A\rangle\rangle$. It is enough to show that mappings of this form are closed under composition. So, we compute $\sigma'\circ\sigma$:

$$\begin{aligned}\sigma'\circ\sigma(\beta)&=\sigma'(S\theta(\beta))\\&=S'\theta'(S\theta(\beta))\quad\text{(because } \sigma' \text{ is continuous and linear)}\\&=S'\theta'(S)(\theta'\circ\theta)(\beta).\end{aligned}$$

Note that $\theta'\circ\theta$ is a continuous algebra endomorphism. Moreover, such an endomorphism, if it maps each letter onto a rational series, preserves rationality. Hence, $\theta'(S)$ is rational, and so is $S'\theta'(S)$, which proves the claim. □

4. Generating functions

We now turn to the commutative images of the power series under consideration, for which we obtain explicit rational expressions. Let the alphabet A have n elements.

Theorem 4.1. *For $d \geqslant 1$ and any word β of length k, one has*

$$\sum_{\alpha\in A^*} \zeta^d(\beta,\alpha)t^{|\alpha|} = \frac{1}{1-nt}\,\frac{1-(n-1)t}{1-(n+n-1)t}\cdots$$

$$\cdots\frac{1-(d-1)(n-1)t}{1-(n+(d-1)(n-1))t}\left(\frac{t}{1-d(n-1)t}\right)^k,$$

and

$$\sum_{\alpha\in A^*} \mu^d(\beta,\alpha)t^{|\alpha|} = \frac{1-t}{1+(n-1)t}\,\frac{1-(1-(n-1))t}{1+2(n-1)t}\cdots$$

$$\cdots\frac{1-(1-(d-1)(n-1))t}{1+d(n-1)t}\left(\frac{t}{1+d(n-1)t}\right)^k.$$

Proof. Let p be the canonical projection $\mathbb{Z}\langle\langle A\rangle\rangle \to \mathbb{Z}[[t]]$ which sends each letter onto t. Define continuous algebra endomorphisms f, g of $\mathbb{Z}[[t]]$ by

$$f(t) = \frac{t}{1-(n-1)t}, \qquad g(t) = \frac{t}{1+(n-1)t}$$

(which are, of course, inverses of each other). Moreover, define continuous linear endomorphisms u, v of $\mathbb{Z}[[t]]$ by

$$u(t^k) = \frac{1}{1-nt}\, f(t^k), \qquad v(t^k) = \frac{1-t}{1+(n-1)t}\, g(t^k).$$

We have for any letter a

$$f\circ p(a) = f(t) = \frac{t}{1-(n-1)t} = t((n-1)t)^*$$

$$= p(a(A-a)^*) \qquad = p\circ\varphi(a),$$

hence, being algebra morphisms, $f\circ p = p\circ\varphi$.

This implies that for any word β of length k,

$$u\circ p(\beta) = u(t^k) = \frac{1}{1-nt}\, f(t^k)$$

$$= p(A^*) f\circ p(\beta)$$

$$= p(A^*) p\circ\varphi(\beta)$$

$$= p(A^*\varphi(\beta)) = p\circ z(\beta)$$

(by Lemma 2.2); hence, $u\circ p = p\circ z$. This, in turn, implies by Lemma 2.1 that

$$\sum_{\alpha} \zeta^d(\beta,\alpha)t^{|\alpha|} = p\circ z^d(\beta) = u^d\circ p(\beta) = u^d(t^k).$$

This last series is easily computed. We have

$$f^d(t)=\frac{t}{1-d(n-1)t}$$

and we show by induction that

$$u^d(t^k)=Sf(S)\dots f^{d-1}(S)f^d(t^k),$$

where $S=1/(1-nt)$. Indeed, this is true for $d=1$. Suppose it is true for d. Then

$$\begin{aligned}u^{d+1}(t^k)&=u(u^d(t^k))\\&=Sf(u^d(t^k))\end{aligned}$$

because u is continuous and linear. Thus, by the inductive hypothesis,

$$u^{d+1}(t^k)=Sf(S)\dots f^d(S)f^{d+1}(t^k),$$

which was to be shown. Now we deduce that

$$\begin{aligned}f^{d-1}(S)&=\frac{1}{1-n\dfrac{t}{1-(d-1)(n-1)t}}\\&=\frac{1-(d-1)(n-1)t}{1-(n+(d-1)(n-1))t}.\end{aligned}$$

This implies the first formula of the theorem.

For the second formula, note first that $p(L_a)=1/(1-(n-1)t)$ and that

$$p(L)=p\left(1+\sum_{a\in A}aL_a\right)=1+\frac{nt}{1-(n-1)t}=\frac{1+t}{1-(n-1)t}.$$

This easily implies, as above, that $g\circ p=p\circ\bar{\psi}$ and $v\circ p=p\circ\bar{m}$, with the notation of the proof of Theorem 1.1 (Section 2). Hence,

$$\sum_{\alpha}\mu^d(\beta,\alpha)t^{|\alpha|}=v^d(t^k)=Tg(T)\dots g^{d-1}(T)g^d(t^k),$$

where $T=(1-t)/(1+(n-1)t)$. Note that

$$g^d(t)=\frac{t}{1+d(n-1)t}$$

because g^d is the inverse of f^d. Then

$$\begin{aligned}g^{d-1}(T)&=\frac{1-\dfrac{t}{1+(d-1)(n-1)t}}{1+(n-1)\dfrac{t}{1+(d-1)(n-1)t}}\\&=\frac{1-(1-(d-1)(n-1))t}{1+d(n-1)t}.\end{aligned}$$

This implies the second formula. □

Note that when $n=2$ (i.e. A has only two letters), many simplifications are possible in the formulas of Theorem 4.1. The $d=1$ cases of these formulas were derived by counting arguments in [2].

Remarks. (1) From Theorem 4.1 it is easy to deduce the explicit rational expressions for the series

$$\sum_{\alpha,\beta\in A^*} \zeta^d(\beta,\alpha)t^{|\alpha|}q^{|\beta|} \quad \text{and} \quad \sum_{\alpha,\beta\in A^*} \mu^d(\beta,\alpha)t^{|\alpha|}q^{|\beta|}.$$

For instance, if we abbreviate the right-hand side of the first formula of Theorem 4.1 by $Q(t)\cdot(R(t))^k$, then

$$\sum_{\alpha,\beta} \zeta^d(\beta,\alpha)t^{|\alpha|}q^{|\beta|}=\frac{Q(t)}{1-nqR(t)}.$$

(2) It follows from the general theory of Cohen–Macaulay posets [6], and the fact that intervals in subword order are Cohen–Macaulay [2], that for fixed $\beta\leqslant\alpha$

$$\sum_{d\geqslant 0} \zeta^{d+1}(\beta,\alpha)s^d=\frac{h(s)}{(1-s)^{|\alpha|-|\beta|+1}},$$

where $h(s)$ is a polynomial of degree $\leqslant|\alpha|-|\beta|-1$ with nonnegative integer coefficients. The coefficients of $h(s)$ have a combinatorial interpretation in terms of the "rank-selected" Möbius function, in particular the degree $|\alpha|-|\beta|-1$ coefficient equals $|\mu(\beta,\alpha)|=\binom{\alpha}{\beta}_n$.

(3) In view of the previous information it seems natural to consider the trivariate generating function

$$\sum_{\substack{\alpha,\beta\in A^*\\ d\geqslant 0}} \zeta^{d+1}(\beta,\alpha)t^{|\alpha|}q^{|\beta|}s^d.$$

Does it have a closed rational expression? One could also consider summing over negative d, i.e. powers of the Möbius function.

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