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Plethysm and conjugation of quasi-symmetric functions

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Abstract

Let F_C denote the basic quasi-symmetric functions, in Gessel's notation (1984) (C any composition). The plethysm $s_\lambda \circ F_C$ is a positive linear combination of functions F_D. Under certain conditions, the image under the involution ω of a quasi-symmetric function defined by equalities and inequalities of the variables is obtained by negating the inequalities. © 1998 Elsevier Science B.V. All rights reserved

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0. Introduction

Quasi-symmetric functions are a generalization of symmetric functions. They appear in [1–4,8–12] in connection with enumeration of permutations, the Robinson–Schensted correspondence, reduced decompositions, (P, ω)-partitions, the descent algebra and noncommutative symmetric functions.

We consider here the λ-ring structure of the ring of quasi-symmetric functions, i.e., the plethysm of a quasi-symmetric functions into a symmetric function. We show that the plethysm $s_\lambda \circ F_C$ is a positive linear combination of F_D's, which are the basic functions defined in [3]. We also study quasi-symmetric functions defined by inequality/equality conditions on the variables, and give a condition which ensures that the conjugate (image under the involution ω) of these functions is obtained by reversing the inequalities, and exchanging strict and large inequalities (a well-known phenomenon for Schur functions).

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The proofs use the theory of (P,ω)-partitions, together with a generalization of it, and a result of [5], expressing the lexicographic order without using equality.

1. Quasi-symmetric functions

The ring *Q*Sym of quasi-symmetric functions is the free $\mathbb{Z}$-module over the functions $M_C \in \mathbb{Z}[[X]]$, X a totally ordered infinite set of commuting varibales, defined for any composition $C=(c_1,\ldots,c_k)$ by

$$M_C = \sum_{x_1<\cdots<x_k} x_1^{c_1}\cdots x_k^{c_k}.$$

*Q*Sym has another basis (F_C), related to (M_C) by

$$F_C = \sum_D M_D, \tag{1.1}$$

where the sum is over all compositions D which are finer than C, e.g., $F_{21}=M_{21}+M_{111}$. These functions are also defined by the formula

$$F_C = \sum x_1\cdots x_n$$

where the sum is subject to the conditions $x_i \leqslant x_{i+1}$, and $x_i < x_{i+1}$ if $i\in S$, the subset of $\{1,\ldots,n-1\}$ associated to C. For these results, see [3]. Note that in [2], the M_C are called *quasi-monomial functions* and the F_C *quasi-ribbon functions*.

2. Plethysm

The ring $\mathbb{Z}[[X]]$ is a λ-ring, where the Adams operators ψ_l are the continuous ring endomorphisms of $\mathbb{Z}[[X]]$ defined by $\psi_l(x)=x^l$ for all x in X. Then clearly $\psi_l(M_C)=M_{lC}$, where $lC=(lc_1,\ldots,lc_k)$. Hence *Q*Sym is a sub-λ-ring. If g is any symmetric function and F any quasi-symmetric function, we may thus define $g\circ F$, as in [6]. The reader who does not like λ-rings may proceed to the next paragraph, where we define directly $g\circ F$, when F is a sum of monomials: this is the only case that we use in Theorem 2.1.

If $F=\sum_{i\in I} m_i(*)$ is written as a sum of monomials, then $g\circ F = g(m_i,\ i\in I)$, i.e. $g\circ F$ is obtained by replacing the variables of g by the monomials m_i; this classical result may be seen as follows: the mappings $g\mapsto g\circ F$ and $g\mapsto g(m_i,\ i\in I)$ are both algebra homomorphisms of the ring of symmetric functions into *Q*Sym. For $g=p_l$, the lth power sum, one has $p_l\circ F=\psi_l(F)=F(x^l, x\in X)=\sum_{i\in I} m_i^l = p_l(m_i,\ i\in I)$, so that both endomorphisms coincide on p_l. Now, the p_l generate the ring of symmetric functions, which implies the equality in general (one has to work over $\mathbb{Q}$).

Observe that since g is symmetric, the order chosen in the sum $(*)$ is immaterial. It is this operation which we may call *plethysm*.

It is a classical result that for two Schur functions s_λ and s_μ, the plethysm $s_\lambda \circ s_\mu$ is a sum of Schur functions; see [7]. Since the functions F_C play, mutatis mutandis, the same role in the theory of quasi-symmetric functions and (P,ω)-partitions that the Schur functions play in the theory of symmetric functions and tableaux, the following result solves a natural question about this plethysm.

Theorem 2.1. *The quasi-symmetric function $s_\lambda \circ F_C$ is a sum of functions F_D.*

By standard formulas in λ-rings, this implies that $g \circ F$ is a sum of functions F_D, if F is a sum of functions F_C and if g is a sum of Schur functions.

Let G be a finite directed graph, with simple edges; let the set E of edges be partitioned into two disjoint subsets E_s and E_w, and call an edge in E_s (resp. E_w) *strict* (resp. *weak*). A *G-partition* is a function $f: V \to X$ such that for any vertices v, v' in V, one has $f(v) \leqslant f(v')$ (resp. $f(v) < f(v')$) if (v,v') is a weak (resp. strict) edge. Then, we define the quasi-symmetric function

$$\Gamma(G) = \sum_f \prod_{v \in V} f(v), \tag{2.1}$$

where the summation is over all G-partitions f.

To such a graph G, associate the graph G' obtained by reverting the strict edges.

Lemma 2.2. *If G and G' are acyclic, then $\Gamma(G)$ is a sum of F_C's.*

Proof. Since G is acyclic, there is a partial order $\leqslant_P$ on V, which is generated by the relations $v \leqslant_P v'$, $(v,v') \in E$, and which turns V into a poset P. Similarly, there is another partial order on V, generated by the edges of the graph G', and which may be extended into a linear order on V. Thus, there is a bijection $\omega: V \to \{1,\ldots,n\}$ such that: $(v,v') \in E_w \Rightarrow \omega(v) < \omega(v')$, and $(v,v') \in E_s \Rightarrow \omega(v) > \omega(v')$.

Now, $V = P$ is a labelled poset. Recall that a (P,ω)-partition is a function $f: P \to X$ such that if $p \leqslant_P q$ then $f(p) \leqslant f(q)$, and if moreover $\omega(p) > \omega(q)$, then $f(p) < f(q)$. We verify that P-ω-partitions and G-partitions coincide.

Let f be a P-ω-partition. If (v,v') is a weak edge, then $v \leqslant_P v'$, hence $f(v) \leqslant f(v')$. If (v,v') is a strict edge, then $\omega(v) > \omega(v')$, and $v \leqslant_P v'$; thus $f(v) < f(v')$. This shows that f is a G-partition. Conversely, if f is a G-partition, suppose that $p \leqslant_P q$. Then, by construction of $\leqslant_P$, there is a chain of vertices $p = v_0, v_1, \ldots, v_n = q$ such that each (v_i, v_{i+1}) is an edge in G. Then $f(v_i) \leqslant f(v_{i+1})$, hence $f(p) \leqslant f(q)$. If moreover $\omega(p) > \omega(q)$, then we cannot have $\omega(v_i) < \omega(v_{i+1})$ for each i, which implies that the edges (v_i, v_{i+1}) are not all weak; hence, some (v_i, v_{i+1}) is strict and $f(v_i) < f(v_{i+1})$, and finally $f(p) < f(q)$.

Now, by a result of Stanley [10] (see also [3]), the quasi-symmetric generating function of (P,ω), i.e the right-hand side of (2.1), where the summation if over all P–ω-partitions f, is equal to $\sum_\alpha F_{C(\alpha)}$, where the summation is over all linear extensions α of the poset P, and where $C(\alpha)$ is the descent composition of the corresponding permutation. The lemma follows. □

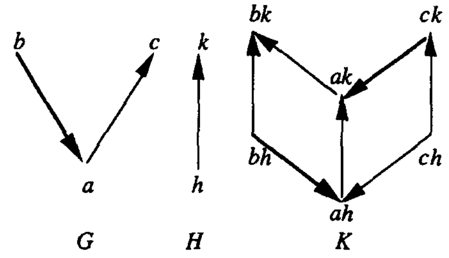


Fig. 1.

Let G,H be graphs as before, with $G=(V,E)$, $H=(W,F)$. Consider all graphs K with set of vertices $V\times W$ and edges satisfying: there is a weak (resp. strict) edge from (v,w) to (v,w') in K if (w,w') is a weak (resp. strict) edge in H; there is an edge from (v,w) to (v',w) or from (v',w) to (v,w), which may be weak or strict, if there is an edge from v to v' in G. See Fig. 1 for an example of such graphs G,H and K. Strict edges are bold.

Lemma 2.3. *If the undirected graph underlying G is a tree and if H,H' are acyclic, then the graphs K and K' are acyclic.*

Proof. Suppose there is a closed path in $K:(v_0,w_0)\to(v_1,w_1)\to\cdots\to(v_n,w_n)=(v_0,w_0)$, where the (v_i,w_i) are distinct for $i=0,\dots,n-1$. Then for each i, either $v_i=v_{i+1}$ or $w_i=w_{i+1}$; in the first case, there is an edge $w_i\to w_{i+1}$ in H.

Hence, there is a closed path in H, except if all w_i are equal. In this case, we have a path in the undirected graph underlying $G: v_0,v_1,\dots,v_n=v_0$, and the v_i are distinct for $i=0,\dots,n-1$. Since G is a tree, we must have $n=0$. Hence, there is no closed path in K.

For K', observe that it is obtained from G and H', exactly as K was obtained from G and H. This shows that K' is acyclic. □

Let A,B be totally ordered sets. Order $A\times B$ lexicographically, that is

$$(a,b)<(a',b') \Leftrightarrow a<a' \text{ or } (a=a' \text{ and } b<b').$$

A fundamental observation of Gordon [5] is that the weak and strict lexicographical order may be defined without using the symbol =. Indeed

$$(a,b)<(a',b') \Leftrightarrow (a<a' \text{ and } b\geqslant b') \text{ or } (a\leqslant a' \text{ and } b<b')$$

and

$$(a,b)\leqslant(a',b') \Leftrightarrow (a\leqslant a' \text{ and } b\leqslant b') \text{ or } (a<a' \text{ and } b>b').$$

Observe that the two cases in both right-hand sides are mutually exclusive, since so are the conditions on b and b'.

The lexicographic order on A^n is defined recursively. Then the previous observations imply the following lemma.

Lemma 2.4. *There exist 2^n sequences $(R_1,\dots,R_n)$, with each R_i in $\{<,\leqslant,>,\geqslant\}$, such that the condition $(a_1,\dots,a_n)<(b_1,\dots,b_n)$ (resp. $(a_1,\dots,a_n)\leqslant(b_1,\dots,b_n)$) is equivalent to the disjoint union of the 2^n conditions:*

$$a_1R_1b_1 \text{ and } a_2R_2b_2 \text{ and} \dots \text{and } a_nR_nb_n. \tag{2.2}$$

Proof of Theorem 2.1. (1) Let m_i, $i\in I$, be a family of totally ordered monomials. Then for any quasi-symmetric function F, the function $F(m_i,\ i\in I)$ is well-defined. Take as a family of monomials those appearing in the function F_D (which is multiplicity-free by (1.1)). Then $s_\lambda\circ F_D=s_\lambda(m_i,\ i\in I)$. Since s_λ is a sum of F_C [3,10,12], it is enough to show that $F_C(m_i,\ i\in I)$ is a sum of F_E's. We order monomials of equal degree, written as an increasing product of variables, by lexicographic order.

Then denote $F_C\circ F_D=F_C(m_i,\ i\in I)$.

(2) There exist graphs G and H, whose underlying undirected graphs are paths such that $\Gamma(G)=F_C$, $\Gamma(H)=F_D$. Indeed, we may take $W=\{1,\dots,n\}$, with $(i,i+1)$ a weak (resp. strict) edge in H if $i\notin S$ (resp. $i\in S$), where S is the subset of $\{1,\dots,n-1\}$ associated to the composition D.

Then $F_D=\sum_f f(1)\dots f(n)$, where the sum is over all H-partitions f.

(3) Order the H-partitions by lexicographic order: $f\leqslant g$ *if* $(f(1),\dots,f(n))\leqslant(g(1),\dots,g(n))$ in lexicographic order. Then $F_C\circ F_D=F_C(f_i(1)\dots f_i(n))$, $i\in I$, where $f_i, i\in I$, are these H-partitions in order.

Since by Lemma 2.4, the lexicographic order is a disjoint union of relations of the form (2.2), we deduce that $F_C\circ F_D$ is a sum of functions $\Gamma(K)$, where K is obtained as in Lemma 2.3. By Lemma 2.2 this implies that $\Gamma(K)$ is a sum of F_E's and concludes the proof. □

We illustrate the proof of Theorem 2.1 by the computation of $F_{21}\circ F_2$ (with the notations of the latter proof). We have $F_{21}=\Gamma(G)$ and $F_2=\Gamma(H)$, where G and H are shown in Fig. 2.

By using the equations before Lemma 2.4, we find that $F_{21}\circ F_2$ is the sum of the $\Gamma(K)$ for K being each of the four graphs shown in Fig. 3.

Indeed, we have $F_{21}\circ F_2=\sum a_1b_1a_2b_2a_3b_3$ where the sum is over all a_1,a_2,a_3,b_1,b_2,b_3 in X such that $a_i\leqslant b_i$ and $(a_1,b_1)\leqslant(a_2,b_2)<(a_3,b_3)$. But the latter condition is equivalent to $((a_1\leqslant a_2$ and $b_1\leqslant b_2)$ or $(a_1<a_2$ and $b_1>b_2))$ and $((a_2<a_3$ and $b_2\geqslant b_3)$ or $(a_2\leqslant a_3$ and $b_2<b_3))$, which in turn is equivalent to the (disjoint) union of the four conditions

$$(a_1\leqslant a_2 \text{ and } b_1\leqslant b_2 \text{ and } a_2<a_3 \text{ and } b_2\geqslant b_3)$$

or

$$(a_1\leqslant a_2 \text{ and } b_1\leqslant b_2 \text{ and } a_2\leqslant a_3 \text{ and } b_2<b_3)$$

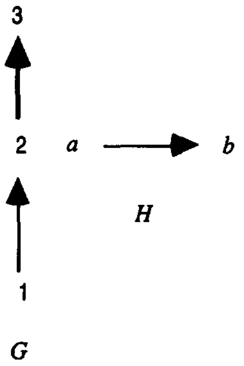


Fig. 2.

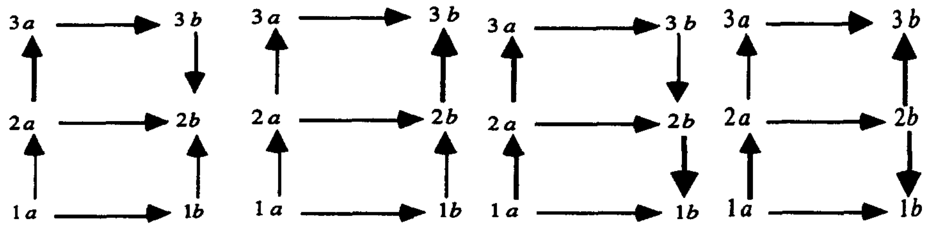


Fig. 3.

or

$$(a_1 < a_2 \text{ and } b_1 > b_2 \text{ and } a_2 < a_3 \text{ and } b_2 \geqslant b_3)$$

or

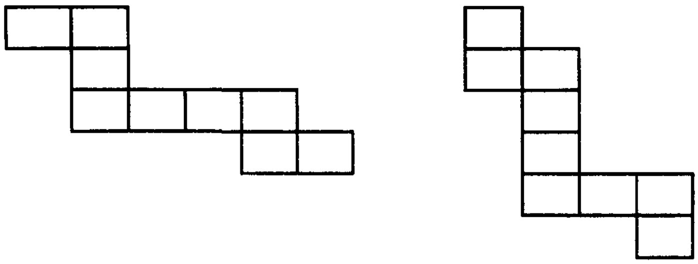
$$(a_1 < a_2 \text{ and } b_1 > b_2 \text{ and } a_2 \leqslant a_3 \text{ and } b_2 < b_3),$$

corresponding to the four graphs in Fig. 3.

3. Conjugation

It is well-known that if s_λ is a Schur function, then $\omega(s_\lambda)$, the *conjugate* of s_λ, with the notations of [7], is obtained from s_λ by interchanging strict and large inequalities in the combinatorial definition of s_λ. For example, if $\lambda = 32$, we have $s_\lambda = \sum abcde$, where the summation condition is $a \leqslant b \leqslant c$, $d \leqslant e$, $a < d$, $b < e$; next, $\omega(s_\lambda) = s_{\lambda'} = s_{221} = \sum abcde$, where the condition is $a < b < c$, $d < e$, $a \leqslant d$, $b \leqslant e$.

Note that, since s_λ is symmetric, the previous condition may be replaced by $a > b > c$, $d > e$, $a \geqslant d$, $b \geqslant e$. We say that this condition is obtained from the first by *conjugation* (i.e. replace $<$ by $\geqslant$ and $\leqslant$ by $>$).



$n = 9$

$C = (2,1,4,2)$ $\quad\quad$ $\omega(C) = (1,2,1,1,3,1)$

$I(C) = \{2,3,7\}$ $\quad\quad$ $\tilde{I}(\omega(C)) = \{8,6,5,4,1\}$

Fig. 4.

Note that the notation ω here has nothing to do with the ω in (P,ω)-partitions. We apologize for this possible ambiguity.

We extend this to quasi-symmetric functions. Define $\omega : Q\mathrm{Sym} \to Q\mathrm{Sym}$ by

$$\omega(F_C) = F_{\omega(C)}, \tag{3.1}$$

where $\omega(C)$ is the composition defined by: $I(C)$ and $\tilde{I}(\omega(C))$ are complementary subsets of $\{1,\dots,n-1\}$, where $|C| = n$, $I(C)$ is $\{c_1, c_1+c_2, \dots, c_1+\cdots+c_{k-1}\}$ if $C = (c_1,\dots,c_k)$, $\tilde{I}(C) = I(\tilde{C})$ and $\tilde{C}$ the reverse of C. Equivalently, C and $\omega(C)$, when represented by skew shapes, are transpose each of another. See Fig. 4.

It has been shown by Gessel (1990, unpublished manuscript; see also [1,8]) that ω is an involutive antomorphism of QSym, extending the classical automorphism ω of the ring of symmetric functions [7].

We say that a quasi-symmetric function F is *defined by a set of equality and inequality conditions* if $F = \sum x_1 \dots x_n$, where the summation is over all x_i's in X satisfying a set of conditions, each of the form $x_i R x_j$, with $R \in \{<, \leqslant, >, \geqslant, =\}$ (the set depends only on F).

For example, each Schur function, each F_C or M_C is of this form (e.g. M_{21} is defined by the conditions $x_1 = x_2$, $x_2 < x_3$). The *sign* of the set of conditions is $(-1)^k$, where k is the number of equalities in the set. The *conjugate* of the set is obtained, as above, by replacing each $x_i < x_j$ by $x_i \geqslant x_j$ and $x_i \leqslant x_j$ by $x_i > x_j$.

Let C be as above a set of conditions on the variables $x_1,\dots,x_n$. We define two graphs, with directed and undirected edges, with vertices $1,2,\dots,n$, as follows: there is an undirected edge $i - j$ in G and G' if $x_i = x_j$ is in C, and a directed edge $i \to j$ in G (resp. G') if $x_i \leqslant x_j$ or $x_i < x_j$ (resp. if $x_i \leqslant x_j$ or $x_i > x_j$) is in C.

We say that such a graph is *acyclic* if there is no closed simple path in it, where a path is a compatible sequence of edges (such a graph looks like the streets in a city, with one and two-way streets); the path $i - j - i$ $(i \neq j)$ is not considered as a simple closed path.

Theorem 3.1. *Let C be a set of equalities and inequalities, F its associated quasi-symmetric function, and $(-1)^k$ its sign. If the graphs G, G' defined above are acyclic, then $(-1)^k\omega(F)$ is defined by the conjugate set.*

Remark. The reader may verify that the condition of acyclicity implies that for each $i \neq j$, one has at most one inequality or equality between x_i and x_j in C.

Examples. (1) By Fig. 1, $\omega(F_{2142}) = F_{121131}$, which are, respectively, defined by the conditions $x_1 \leqslant x_2 < x_3 \leqslant x_4 \leqslant x_5 \leqslant x_6 \leqslant x_7 < x_8 \leqslant x_9$ and $x_9 < x_8 \leqslant x_7 < x_6 < x_5 < x_4 \leqslant x_3 \leqslant x_2 < x_1$.

(2) By [7], $\omega(p_k) = (-1)^{k-1} p_k$, and p_k is defined by the conditions $x_1 = x_2 = \cdots = x_k$.

(3) More generally, by [1,9], $\omega(M_C) = (-1)^{|C|-\ell(C)} \sum_D M_{\tilde{D}}$, where the summation is over all compositions D which are less fine than C, and $\tilde{D}$ is the reversal of D. For example, $\omega(M_{231}) = (-1)^{6-3}(M_{132} + M_{42} + M_{15} + M_6)$, which may be written

$$\begin{aligned}
\omega\left(\sum_{a=b<c=d=e<f} abcdef\right) &= -\sum_{x<y=z=t<u=v} xyztuv - \sum_{x=y=z=t<u=v} xyztuv \\
&\quad - \sum_{x<y=z=t=u=v} xyztuv - \sum_{x=y=z=t=u=v} xyztuv \\
&= -\sum_{x \leqslant y=z=t \leqslant u=v} xyztuv \\
&= -\sum_{a=b \geqslant c=d=e \geqslant f} abcdef.
\end{aligned}$$

(4) The theorem applies to all inequality conditions defined by graphs G satisfying the hypothesis of Lemma 2.2. In particular, to $P-\omega$-partitions and Young diagrams.

We use again the definitions of Section 2.

Lemma 3.2. *Let G be a directed graph, with weak and strict edges. Let $\omega(G)$ be the graph obtained by reversing the edges and exchanging strict and weak edges. If G and G' are acyclic, then $\Gamma(\omega(G)) = \omega(\Gamma(G))$.*

Proof. We use the proof of Lemma 2.2, and conclude that $\Gamma(G) = \sum_\alpha F_{C(\alpha)}$, where the sum is over all linear extensions of P.

Similarly, taking the reverse poset with the same labelling, we find that $\Gamma(\omega(G)) = \sum_\alpha F_{C(\tilde{\alpha})}$, with the same summation condition, where $\tilde{\alpha}$ is the reversal of α. Now, $C(\tilde{\alpha}) = \omega(C(\alpha))$, hence (3.1) implies that $\Gamma(\omega(G)) = \omega(\Gamma(G))$. □

Proof of Theorem 3.1 (*Induction on the number k of equalities*). (1) If $k = 0$, then $F = \Gamma(G)$, with the notations of (2.1), where the edges of G corresponding to weak (resp. strict) inequalities are weak (resp. strict.).

Then the graph of the conjugate set of C is $\omega(G)$, obtained as in Lemma 3.2. Thus, the theorem follows in this case.

(2) Suppose now that there is an equality $x_i = x_j$ in C. We define two sets of equalities and inequalities, C_1 and C_2, by replacing $x_i = x_j$ by $x_i \leqslant x_j$ and $x_i < x_j$ respectively. Let F_1, F_2 be the corresponding functions. Then $F = F_1 - F_2$. Now, the acyclicity of the graphs G, G' implies that of G_1, G_1', G_2, G_2'. Hence, by induction, $(-1)^{k-1}\omega(F_1)$ and $(-1)^{k-1}\omega(F_2)$ are defined by the sets of conditions $\omega(C_1)$ and $\omega(C_2)$ respectively. Now, these sets are obtained from $\omega(C)$ by replacing in it $x_i = x_j$ by $x_i > x_j$ and $x_i \geqslant x_j$. Hence the functions F', F_1', F_2' corresponding to $\omega(C), \omega(C_1), \omega(C_2)$ satisfy $F' = F_2' - F_1'$. Since, as we saw, $\omega(F_1) = (-1)^{k-1}F_1'$, $\omega(F_2) = (-1)^{k-1}F_2'$, we obtain $\omega(F) = \omega(F_1) - \omega(F_2) = (-1)^{k-1}(F_1' - F_2') = (-1)^k F'$, which is what was to be shown.

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