Discrete Comput Geom
DOI 10.1007/s00454-015-9681-2

A d-dimensional Extension of Christoffel Words

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Received: 22 April 2014 / Revised: 16 February 2015 / Accepted: 25 February 2015

Abstract In this article, we extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in an arbitrary dimension that we call Christoffel graphs. Christoffel graphs, when $d = 2$, correspond to the well-known Christoffel words. Due to periodicity, the d-dimensional Christoffel graph can be embedded in a $(d - 1)$-torus (a parallelogram when $d = 3$). We show that Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part and conjugation with their reversal. Our main result extends Pirillo's theorem (characterization of Christoffel words which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma) to an arbitrary dimension. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is a translation. We show that a fully periodic subgraph of the hypercubic lattice is a translation of its flip if and only if it is a Christoffel graph.

Keywords Christoffel words · Christoffel graphs · Digital hyperplane · Flip · Hypercubic lattice · Pirillo's theorem

Mathematics Subject Classification 05C75 · 52C35 · 68R15

Editor in Charge: János Pach

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Published online: 24 April 2015

1 Introduction

This article is a contribution to the study of digital planes and hyperplanes in any dimension d. We study only rational hyperplanes, that is, those which are defined by an equation with rational coefficients. We extract from such a hyperplane a finite pattern that we call, for $d = 3$, a *Christoffel parallelogram*. We show that Christoffel parallelograms are a generalization of Christoffel words.

Digital planes were introduced in [25] and further studied in [1,16,19,27]. Recognition algorithms were proposed in [20,24,26]. See [12] for a complete review about many aspects of digital planarity, such as characterizations in arithmetic geometry, periodicity, connectivity, and algorithms. Digital planes can be seen as unions of square faces. Such stepped surfaces, introduced in [21,22] as a way to construct quasiperiodic tilings of the plane, can be generated from multi-dimensional continued fraction algorithms by introducing substitutions on square faces [2,3].

While digital planes are a satisfactory generalization of Sturmian words, it is still unclear as to what is the equivalent notion of Christoffel words in a higher dimension. In [18, Figs. 6.6 and 6.7], fundamental domains of rational digital planes are constructed from the iteration of generalized substitutions on the unit cube. Recently, Domenjoud and Vuillon [17] generalized central words to arbitrary dimension using palindromic closure. In both cases, the representation is nonconvex and has a fractal-like boundary.

In this article, we propose to extend the definition of Christoffel words to directed subgraphs of the hypercubic lattice in an arbitrary dimension that we call Christoffel graphs. A similar construction, called *roundwalk*, but serving a different purpose, was given in [7], producing multi-dimensional words that are closely related to k-dimensional Sturmian words. Christoffel graphs, when $d = 2$, correspond to Christoffel words. Due to its periods, the d-dimensional Christoffel graph can be embedded in a $(d-1)$-torus, and when $d = 3$, the torus is a parallelogram. This extension is motivated by Pirillo's theorem which asserts that a word amb is a Christoffel word if and only if it is conjugate to bma. In the generalization, the map $amb \mapsto bma$ is seen as a flip operation on graphs embedded in $\mathbb{Z}^d$ and the conjugation is replaced by some translation. When $d = 3$, our flip corresponds to a flip in a rhombus tiling [4,9,10]. We show that these Christoffel graphs have similar properties to those of Christoffel words: symmetry of their central part (Lemma 11) and conjugation with their reversal (Corollaries 2 and 3). Our main result is Theorem 3, which extends Pirillo's theorem in an arbitrary dimension.

We recall in Sect. 2 the basic notions related to Christoffel words and digital planes. The digital hyperplane graphs are defined in Sect. 3. The operations on them (flip, reversal and translation) are introduced in Sect. 4. We show that the flip of a Christoffel graph is a translation of itself in Sect. 5. This is the sufficiency of the Pirillo theorem. In Sect. 6, we consider the necessity and obtain a d-dimensional Pirillo theorem, our main result. Finally, we construct in the appendix the mathematical framework for the definition of digital hyperplanes, since we could not find explicit and complete references.

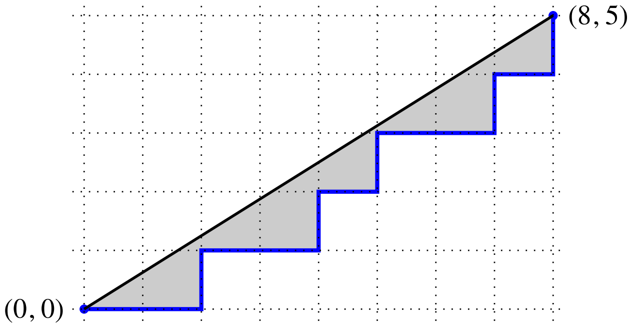


Fig. 1 The lower Christoffel word $w = aabaababaabab$

2 Christoffel Words and Digital Planes

2.1 Christoffel Words

Christoffel words are obtained by discretizing a line segment in the plane as follows: let $(p, q) \in \mathbb{N}^2$ with $\gcd(p, q) = 1$, and let S be the line segment with endpoints (0, 0) and (p, q). The word $w \in \{a, b\}^*$ is a *lower Christoffel word* if the path induced by w starting at the origin ends at (p, q), is under S, and the path and the segment S delimit a polygon with no integral interior point. An *upper Christoffel word* is defined similarly by taking the path which is above the segment. A *Christoffel word* is a lower Christoffel word. See Fig. 1 and [6]. An astonishing result about Christoffel words is the following characteristic property given by Pirillo [23]. Recall that two words w and w' are *conjugate* if there exist two words u and v such that $w = uv$ and $w' = vu$.

Theorem 1 (Pirillo) *A word $w = amb \in \{a, b\}^*$ is a Christoffel word if and only if amb and bma are conjugate.*

It is also known that the two words amb and bma are conjugate by palindromes [15, Thm. 3.1] (see also [11, Prop. 6.1]): for example, the Christoffel word in Fig. 1 can be factorized as a product of two palindromes, but also as a letter (a), a *central word* m, and a last letter (b):

$$w = aabaa \cdot babaabab = a \cdot abaababaaba \cdot b = a \cdot m \cdot b,$$

and the conjugate word w' of w obtained by exchanging the two palindromes can also be factorized as the product of a letter (b), the same central word m, and a last letter (a):

$$w' = babaabab \cdot aabaa = b \cdot abaababaaba \cdot a = b \cdot m \cdot a.$$

Central words are the words m such that amb is a Christoffel word. They can be defined independently of Christoffel words: a word m is a *central word* if and only if for some coprime integers p and q, the length of m is $p + q - 2$ and p and q are periods of m.

In this case, the Christoffel word amb is associated as above to the vector (p,q). See [14] for more information and [5] for 14 different characterizations of central words. There are also some properties, which are satisfied by Christoffel words, but do not characterize them.

Lemma 1 *Let $w = amb$ be a Christoffel word of vector (p,q). Then*

(i) *the central word m is a palindrome: $\widetilde{m} = m$;*
(ii) *p is a period of am and q is a period of mb;*
(iii) *the reversal $\widetilde{w}$ of w is conjugate to w.*

The proof of (iii) follows from (i) and from Theorem 1. Words conjugate to their reversal were studied in [13], and are the product of two palindromes and are not necessarily Christoffel words. Moreover, not every palindrome is a central word. In this article, we generalize Theorem 1 to dimension $d \geq 3$. We also show that properties like the one enumerated in Lemma 1 hold.

2.2 Digital Planes

Given $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $\mu, \omega \in \mathbb{R}$, the *lower arithmetical digital plane* [25] $\mathcal{P}$ of normal vector $\mathbf{a}$ is the set of points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ satisfying

$$\mu \leq a_1x_1 + a_2x_2 + a_3x_3 < \mu + \omega.$$

The parameter ω is called the (arithmetic) *width*. If $\omega = \|\mathbf{a}\|_1 = |a_1| + |a_2| + |a_3|$, then the digital plane is said to be *standard*. Standard arithmetical digital planes can be furnished with a canonical structure of a two-dimensional, connected, orientable combinatorial manifold without boundary, whose faces are quadrangles and whose vertices are points on the plane [19]. See the appendix, where we provide the mathematical framework for the definition of digital hyperplanes $\mathcal{P}$ and *stepped surfaces* $\mathcal{S}$ [4].

Let k be an integer such that $1 \leq k \leq d$. We say that $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ are *k-neighbors* if and only if

$$\|\mathbf{v} - \mathbf{u}\|_\infty = 1 \quad \text{and} \quad \|\mathbf{v} - \mathbf{u}\|_1 \leq k.$$

Note that $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$ are 1-neighbors if and only if their difference is $\pm\mathbf{e}_i$ for some i such that $1 \leq i \leq d$. In this article, we are interested in the graph representing the 1-neighboring relation for the digital plane $\mathcal{P}$ and in general the 1-neighboring relation for a digital hyperplane in $\mathbb{Z}^d$.

3 Digital Hyperplane Graphs

Let $a_1, \ldots, a_d$ be relatively prime positive integers and $s = \|\mathbf{a}\|_1 = \sum a_i$ be their sum. We denote $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{N}^d$. We define the mapping $\mathcal{F}_\mathbf{a} : \mathbb{Z}^d \to \mathbb{Z}/s\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$ onto $\sum_i a_i x_i \bmod s$. We identify $\mathbb{Z}/s\mathbb{Z}$ and

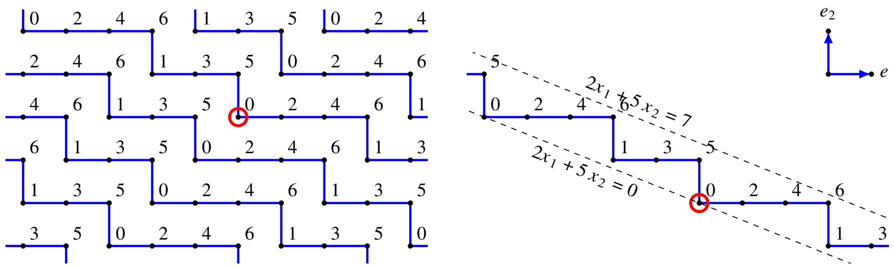


Fig. 2 *Left* The graph $\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$. *Right* Standard digital line $\mathcal{P}$ of normal vector $\mathbf{a} = (2, 5)$

$\{0, 1, \ldots, s-1\}$. A total order on $\mathbb{Z}/s\mathbb{Z}$ is defined correspondingly; it is this order that is used in the definition of $\mathcal{H}_{\mathbf{a}}$ below. The map $\mathcal{F}_{\mathbf{a}}$ induces a $\mathbb{Z}^d$-action $\mathbf{x} \cdot g = g + \mathcal{F}_{\mathbf{a}}(\mathbf{x})$ on the cyclic group $\mathbb{Z}/s\mathbb{Z}$, so that it is a rational case of the $\mathbb{Z}^2$-action on the torus as studied in [3,8]. We consider $\mathbb{E}_d = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) : \mathbf{u} \in \mathbb{Z}^d \text{ and } 1 \le i \le d\}$, the set of oriented edges of the hypercubic lattice. Note that the set $\mathbb{E}_d$ also corresponds to the Cayley graph of $\mathbb{Z}^d$ with generators $\mathbf{e}_i$ for all i with $1 \le i \le d$.

3.1 The Christoffel Graph $\mathcal{H}_{\mathbf{a}}$

The *Christoffel graph* $\mathcal{H}_{\mathbf{a}}$ of normal vector $\mathbf{a}$ is the subset of edges of $\mathbb{E}_d$ increasing for the function $\mathcal{F}_{\mathbf{a}}$:

$$\mathcal{H}_{\mathbf{a}} = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) < \mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{e}_i)\}.$$

An example of the graph $\mathcal{H}_{\mathbf{a}}$ when $d = 2$ and $\mathbf{a} = (a_1, a_2) = (2, 5)$ is shown in Fig. 2 (left) where the edges are represented in blue and a small red circle surrounds the origin.

The first observation is stated in the next lemma.

Lemma 2 *The graph $\mathcal{H}_{\mathbf{a}}$ is invariant under the translation by the vector $\sum_{i=1}^{d} \mathbf{e}_i = (1, 1, \ldots, 1)$.*

The proof is postponed until Lemma 7, where we show that the graph $\mathcal{H}_{\mathbf{a}}$ is invariant under all translations $\mathbf{t} \in \operatorname{Ker} \mathcal{F}_{\mathbf{a}}$. Because of this invariance, the question is to find a good representative for the equivalence class $\mathbf{x} + (1, 1, \ldots, 1)\mathbb{Z}$ for each $\mathbf{x} \in \mathbb{Z}^d$. It is natural to choose $\bar{\mathbf{x}} \in \mathbf{x} + (1, 1, \ldots, 1)\mathbb{Z}$ such that

$$0 \le \sum a_i \bar{x}_i < s. \tag{1}$$

If $(\mathbf{u}, \mathbf{v})$ is an edge of $\mathcal{H}_{\mathbf{a}}$ such that $\mathbf{v} - \mathbf{u} = \mathbf{e}_i$, then $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ is a pair of points, which are 1-neighbors satisfying (1) and $\bar{\mathbf{v}} - \bar{\mathbf{u}} = \mathbf{e}_i$. Thus, the vertices satisfying (1) are a set of representatives for the vertices of $\mathcal{H}_{\mathbf{a}}$, see Fig. 2 (right). Thus, each connected component of the graph $\mathcal{H}_{\mathbf{a}}$ corresponds exactly to a *standard digital plane* $\mathcal{P}$ with the 1-neighbor relation. The advantage of $\mathcal{H}_{\mathbf{a}}$ over the digital hyperplane $\mathcal{P}$ is its algebraic structure. The next lemma gives an equivalent definition of the edges of the graph $\mathcal{H}_{\mathbf{a}}$. It will be useful in the sequel.

Let $a, b \in [0, s[$ be two integers. If $a < b$, then $]a, b]$ is a subinterval of $[0, s[$. If $a > b$, then $]a, b] =]a, s[\cup [0, b]$ is defined as the union of two subintervals of $[0, s[$.

Lemma 3 *Let* $(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d$ *such that* $\mathbf{v} = \mathbf{u} + \mathbf{e}_i$ *for some* $1 \leq i \leq d$. *Then,*

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}} &\iff \mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [0, s - a_i - 1] \iff \mathcal{F}_{\mathbf{a}}(\mathbf{v}) \in [a_i, s - 1] \\ &\iff 0 \notin]\mathcal{F}_{\mathbf{a}}(\mathbf{u}), \mathcal{F}_{\mathbf{a}}(\mathbf{v})], \qquad (2)\\ (\mathbf{u}, \mathbf{v}) \notin \mathcal{H}_{\mathbf{a}} &\iff \mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [s - a_i, s - 1] \iff \mathcal{F}_{\mathbf{a}}(\mathbf{v}) \in [0, a_i - 1] \\ &\iff 0 \in]\mathcal{F}_{\mathbf{a}}(\mathbf{u}), \mathcal{F}_{\mathbf{a}}(\mathbf{v})]. \qquad (3)\end{aligned}$$

For each permutation σ of the set $\{1, 2, \cdots, d\}$, there exists a σ*-path*

$$(\mathbf{u}, \mathbf{u} + \mathbf{e}_{\sigma(1)}), (\mathbf{u} + \mathbf{e}_{\sigma(1)}, \mathbf{u} + \mathbf{e}_{\sigma(1)} + \mathbf{e}_{\sigma(2)}), \ldots, \left(\mathbf{u} + \sum_{i=1}^{d-1} \mathbf{e}_{\sigma(i)}, \mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_{\sigma(i)}\right)$$

made of d edges of $\mathbb{E}_d$ going from the vertex $\mathbf{u} \in \mathbb{Z}^d$ to the vertex $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$.

Lemma 4 *All of the d edges of a* σ*-path but one belong to* $\mathcal{H}_{\mathbf{a}}$.

Proof Each edge $(\mathbf{u} + \sum_{i=1}^{k-1} \mathbf{e}_{\sigma(i)}, \mathbf{u} + \sum_{i=1}^{k} \mathbf{e}_{\sigma(i)})$ of the σ-path is mapped onto an interval $[\mathcal{F}_{\mathbf{a}}(\mathbf{u}) + \sum_{i=1}^{k-1} a_{\sigma(i)}, \mathcal{F}_{\mathbf{a}}(\mathbf{u}) + \sum_{i=1}^{k} a_{\sigma(i)}[$ by the function $\mathcal{F}_{\mathbf{a}}$. Since $\sum_{i=1}^{d} a_{\sigma(i)} = \sum_{i=1}^{d} a_i = s$, those d sets, with $1 \leq k \leq d$, are a partition of $[0, s[$. Therefore, only one of them contains 0. From Lemma 3, only one edge of the σ-path does not belong to $\mathcal{H}_{\mathbf{a}}$. □

Let $R \subseteq \{1, 2, \ldots, d\}$ and $\mathbf{u} \in \mathbb{Z}^d$. A *hypercube graph from vertex* $\mathbf{u}$ *to vertex* $\mathbf{u} + \sum_{i \in R} \mathbf{e}_i$ of dimension $k = \text{Card}\, R$ with $2^{\text{Card}\, R}$ vertices is the subgraph of $\mathbb{E}_d$ defined by

$$\left\{\left(\mathbf{u} + \sum_{i \in P} \mathbf{e}_i, \mathbf{u} + \sum_{i \in Q} \mathbf{e}_i\right) \in \mathbb{E}_d \;\middle|\; P \subset Q \subseteq R \text{ and } \text{Card}\, Q \setminus P = 1\right\}.$$

Each nonedge $(\mathbf{u}, \mathbf{v})$ of $\mathcal{H}_{\mathbf{a}}$ implies the presence of a $(d-1)$-dimensional hypercube graph from vertex $\mathbf{v}$ to vertex $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$ in $\mathcal{H}_{\mathbf{a}}$. When $d = 2$, the hypercube is a single edge; for example, when $\mathbf{a} = (2, 5)$, $(-\mathbf{e}_1, \mathbf{0}) \notin \mathcal{H}_{\mathbf{a}}$ implies that $(\mathbf{0}, \mathbf{e}_2) \in \mathcal{H}_{\mathbf{a}}$. Also, when $d = 3$, the hypercube is a square; for example, when $\mathbf{a} = (2, 3, 5)$, $(-\mathbf{e}_1, \mathbf{0}) \notin \mathcal{H}_{\mathbf{a}}$ implies that $\{(\mathbf{0}, \mathbf{e}_2), (\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3), (\mathbf{0}, \mathbf{e}_3), (\mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3)\} \subset \mathcal{H}_{\mathbf{a}}$. This is proved in the next lemma.

Lemma 5 *If* $(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \setminus \mathcal{H}_{\mathbf{a}}$, *then the* $(d-1)$*-dimensional hypercube graph from vertex* $\mathbf{v}$ *to vertex* $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$ *with* 2^{d-1} *vertices is a subgraph of* $\mathcal{H}_{\mathbf{a}}$.

Proof From Lemma 4, the last $d - 1$ edges of every σ-path starting with the edge $(\mathbf{u}, \mathbf{v})$ and ending in $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$ are in $\mathcal{H}_{\mathbf{a}}$. The set of last $d - 1$ edges of these paths generates a hypercube graph from vertex $\mathbf{v}$ to vertex $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$. □

A line containing some point $\mathbf{x} \in \mathbb{Z}^d$ parallel to $\mathbf{e}_i$ in the hypercubic lattice $\mathbb{E}_d$ is a set

$$L_{\mathbf{x},i} = \{(\mathbf{x} + k\mathbf{e}_i, \mathbf{x} + (k+1)\mathbf{e}_i) : k \in \mathbb{Z}\} \subset \mathbb{E}_d.$$

The intersection $L_{\mathbf{x},i} \cap \mathcal{H}_{\mathbf{a}}$ of such a line with a digital hyperplane graph $\mathcal{H}_{\mathbf{a}}$ is made of consecutive edges and nonedges. The next lemma states that Christoffel words appear in this sequence.

Lemma 6 *The sequence of consecutive edges and nonedges in $L_{\mathbf{x},i} \cap \mathcal{H}_{\mathbf{a}}$ is periodic and the period is a Christoffel word.*

Proof Each subset $L_{\mathbf{x},i} \cap \mathcal{H}_{\mathbf{a}}$ can be described by the subgroup of $\mathbb{Z}/s\mathbb{Z}$ generated by $\mathcal{F}_{\mathbf{a}}(\mathbf{e}_i)$, i.e.,

$$(\mathbf{x} + k\mathbf{e}_i, \mathbf{x} + (k+1)\mathbf{e}_i) \in L_{\mathbf{x},i} \cap \mathcal{H}_{\mathbf{a}} \iff 0 \leq \mathcal{F}_{\mathbf{a}}(\mathbf{x} + k\mathbf{e}_i) < s - a_i.$$

This corresponds to the well-known construction of Christoffel words from the labeling of Cayley graphs of $\mathbb{Z}/s\mathbb{Z}$ with the generator a_i [6, Sect. 1.2 Cayley graph definition]. □

For example, in the digital hyperplane graph $H_{(2,5)}$ shown in Fig. 2, coding an edge by the letter a and a nonedge by letter b, we get the periods *aaabaab* and *abbabbb* for the lines $L_{\mathbf{x},i} \cap \mathcal{H}_{\mathbf{a}}$ for $i = 1, 2$, respectively. Both are Christoffel words.

Definition 1 (*Image*) Let $f : \mathbb{Z}^d \to S$ be a homomorphism of a $\mathbb{Z}$-module. For some subset of edges $X \subseteq \mathbb{E}_d$, we define the image by f of the edges X by

$$f(X) = \{(f(\mathbf{u}), f(\mathbf{v})) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

This definition allows us to define the graphs $I_{\mathbf{a}}$ and $\mathcal{G}_{\mathbf{a}}$ as projections of $\mathcal{H}_{\mathbf{a}}$ in the sections below.

3.2 The Graph $I_{\mathbf{a}}$

Let π be the orthogonal projection from $\mathbb{R}^d$ onto the hyperplane $\mathcal{D}$ of equation $\sum x_i = 0$. Its restriction to the stepped surface $\mathcal{S}$ of the digital plane $\mathcal{P}$ of normal vector $\mathbf{a} \in \mathbb{Z}^d$ is a bijection onto $\mathcal{D}$. It maps $\mathcal{P}$, the integral points in $\mathcal{S}$, onto a lattice L [3, Sect. 2.2] in $\mathcal{D}$ spanned by the vectors $\mathbf{h}_i = \pi(\mathbf{e}_i)$; they satisfy $\sum_i \mathbf{h}_i = 0$. Note that $\pi(\mathbb{Z}^d)$ is also equal to L, since each point in $\mathbb{Z}^d$ is congruent to some point in $\mathcal{P}$ modulo the kernel of the projection. We may identify the set L and $\mathbb{Z}^d/(1, 1, \ldots, 1)\mathbb{Z}$, since two integral points are projected by π onto the same point if and only if their difference is a multiple of the vector $(1, 1, \ldots, 1)$ and since this multiple is necessarily an integral multiple. Since $\mathcal{F}_{\mathbf{a}}(1, 1, \ldots, 1) = 0$, the mapping $\mathcal{F}_{\mathbf{a}}$ induces a mapping $\mathcal{F}'_{\mathbf{a}} : L \to \mathbb{Z}/s\mathbb{Z}$. We have the following commuting diagram:

$$\begin{array}{ccc} \mathbb{Z}^d & \xrightarrow{\quad\mathcal{F}_{\mathbf{a}}\quad} & \mathbb{Z}/s\mathbb{Z} \\ & \searrow^{\pi} \qquad \nearrow_{\mathcal{F}'_{\mathbf{a}}} & \\ & L=\mathbb{Z}^d/(1,1,\ldots,1)\mathbb{Z} & \end{array}$$

We consider the directed graph whose set of edges is $I_{\mathbf{a}} = \pi(\mathcal{H}_{\mathbf{a}})$. The graphs $I_{\mathbf{a}}$ for $\mathbf{a} = (a_1, a_2) = (2, 5)$ and $\mathbf{a} = (a_1, a_2, a_3) = (2, 3, 5)$ are shown in Fig. 3. Note that the orientation of an edge is redundant when $d = 3$, since each edge is oriented as one of the vectors $\mathbf{h}_i$.

Lemma 5 can be seen on $I_{\mathbf{a}}$ when $d = 3$ by the fact that each nonedge is the short diagonal of a rhombus. For example, if $\mathbf{a} = (2, 3, 5)$, then $(-\mathbf{h}_1, \mathbf{0}) \notin I_{\mathbf{a}}$. From the lemma, the paths $(\mathbf{0}, \mathbf{h}_2)$, $(\mathbf{h}_2, \mathbf{h}_2 + \mathbf{h}_3)$ and $(\mathbf{0}, \mathbf{h}_3)$, $(\mathbf{h}_3, \mathbf{h}_2 + \mathbf{h}_3)$ are in $I_{\mathbf{a}}$.

The next proposition is a generalization of the fact that $I_{\mathbf{a}}$ is a tiling of the rhombus when $d = 3$ proved in [3,19]. Indeed, each rhombus is the projection under π of one of three types of squares in $\mathbb{R}^3$. Below, the projection under π of the convex hull of the 2^k vertices of a k-dimensional hypercube graph in $\mathbb{E}_d$ is called a *k-dimensional parallelotope*. The edges of such a parallelotope have equal lengths. When $d = 3$, a $(d-1)$-dimensional parallelotope is a rhombus.

Proposition 1 *The graph $I_{\mathbf{a}}$ produces a tiling of $\mathcal{D}$ by d types of $(d-1)$-dimensional parallelotopes.*

In the following proof, the fractional part of a real number $x \in \mathbb{R}$ is denoted by $\{x\} = x - \lfloor x \rfloor$.

Proof Each real point $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ of the hyperplane $\mathcal{D}$ is contained in a $(d-1)$-simplex with vertices $\{\pi(\mathbf{u}) + \sum_{i=1}^{k} \mathbf{h}_{\sigma(i)} : 0 \leq k \leq d-1\}$ for $\mathbf{u} = (\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor) \in \mathbb{Z}^d$ and permutation σ of $\{1, 2, \ldots, d\}$ such that

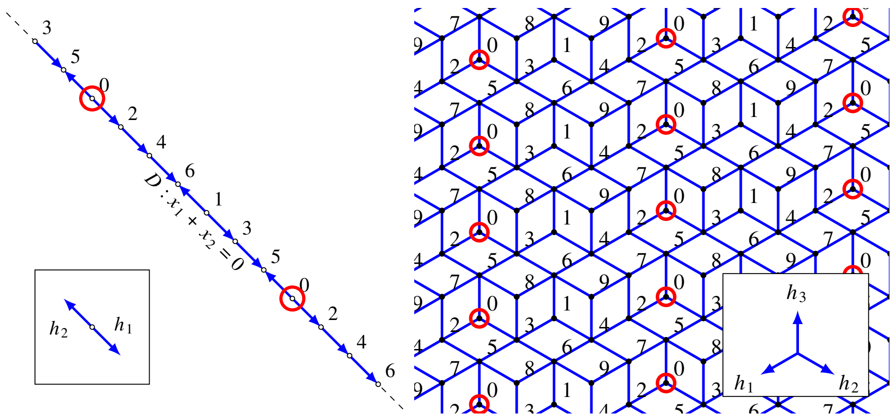


Fig. 3 *Left* The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 5)$. *Right* The graph $I_{\mathbf{a}}$ when $\mathbf{a} = (2, 3, 5)$. The label at each vertex is its image under $\mathcal{F}'_{\mathbf{a}}$

$\{x_{\sigma(1)}\} \geq \{x_{\sigma(2)}\} \geq \cdots \geq \{x_{\sigma(d)}\}$. We illustrate this in an example. Suppose $d = 4$ and $\mathbf{x}$ are such that σ is the identity permutation on $\{1, 2, 3, 4\}$. We have

$$\mathbf{x} = \begin{cases} \mathbf{u} \\ +\{x_1\}\mathbf{e}_1 \\ +\{x_2\}\mathbf{e}_2 \\ +\{x_3\}\mathbf{e}_3 \\ +\{x_4\}\mathbf{e}_4 \end{cases} = \begin{cases} \mathbf{u} \\ +\big(\{x_1\} - \{x_2\}\big)\mathbf{e}_1 \\ +\big(\{x_2\} - \{x_3\}\big)(\mathbf{e}_1 + \mathbf{e}_2) \\ +\big(\{x_3\} - \{x_4\}\big)(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ +\{x_4\}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \end{cases}$$

$$= \begin{cases} \big(1 - \{x_1\}\big)\mathbf{u} \\ +\big(\{x_1\} - \{x_2\}\big)(\mathbf{u} + \mathbf{e}_1) \\ +\big(\{x_2\} - \{x_3\}\big)(\mathbf{u} + \mathbf{e}_1 + \mathbf{e}_2) \\ +\big(\{x_3\} - \{x_4\}\big)(\mathbf{u} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \\ +\{x_4\}(\mathbf{u} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4) \end{cases}$$

Therefore, $\mathbf{x}$ is in the convex hull of the 3-simplex with vertices $\{\pi(\mathbf{u}), \pi(\mathbf{u}) + \mathbf{h}_1, \pi(\mathbf{u}) + \mathbf{h}_1 + \mathbf{h}_2, \pi(\mathbf{u}) + \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3\}$ since

$$\mathbf{x} = \pi(\mathbf{x}) = \begin{cases} (1 + \{x_4\} - \{x_1\})\pi(\mathbf{u}) \\ +(\{x_1\} - \{x_2\})(\pi(\mathbf{u}) + \mathbf{h}_1) \\ +(\{x_2\} - \{x_3\})(\pi(\mathbf{u}) + \mathbf{h}_1 + \mathbf{h}_2) \\ +(\{x_3\} - \{x_4\})(\pi(\mathbf{u}) + \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3) \end{cases}$$

Consider the σ-path in $\mathbb{E}_d$ starting at vertex $\mathbf{u}$ and ending at $\mathbf{u} + \sum_{i=1}^{d} \mathbf{e}_i$. From Lemma 4, there is an edge $(\mathbf{u}', \mathbf{v}')$ of the σ-path that is not in $\mathcal{H}_\mathbf{a}$. From Lemma 5, the hypercube graph from vertex $\mathbf{v}'$ to vertex $\mathbf{u}' + \sum_{i=1}^{d} \mathbf{e}_i$ with 2^{d-1} vertices is a subgraph of $\mathcal{H}_\mathbf{a}$. Therefore, the hypercube graph (projected in $\pi(\mathbb{E}_d)$) going from vertex $\pi(\mathbf{v}')$ to vertex $\pi(\mathbf{u}')$ with 2^{d-1} vertices is a subgraph of $I_\mathbf{a}$. The convex hull of this graph contains the $(d-1)$-simplex with vertices $\{\pi(\mathbf{u}) + \sum_{i=1}^{k} \mathbf{h}_{\sigma(i)} : 0 \leq k \leq d-1\}$, which in turn, contains $\mathbf{x}$. Therefore, the point $\mathbf{x}$ of the hyperplane $\mathcal{D}$ is contained in the image under π of the convex hull of a $(d-1)$-dimensional hypercube graph in $\mathbb{E}_d$. For almost every $\mathbf{x}$, the inequalities $\{x_{\sigma(1)}\} > \{x_{\sigma(2)}\} > \cdots > \{x_{\sigma(d)}\}$ are strict and the parallelotope is unique. We conclude that $\mathcal{D}$ is tiled by $(d-1)$-dimensional parallelotopes. □

3.3 Kernel of $\mathcal{F}_\mathbf{a}$ and $\mathcal{F}'_\mathbf{a}$

Lemma 7 *The digital hyperplane graph $\mathcal{H}_\mathbf{a}$ is invariant under any translation $\mathbf{t} \in \operatorname{Ker} \mathcal{F}_\mathbf{a}$.*

Proof Let $\mathbf{u} \in \mathbb{Z}^d$ and $\mathbf{t} \in \operatorname{Ker} \mathcal{F}_\mathbf{a}$. We have $\mathcal{F}_\mathbf{a}(\mathbf{u} + \mathbf{t}) = \mathcal{F}_\mathbf{a}(\mathbf{u}) + \mathcal{F}_\mathbf{a}(\mathbf{t}) = \mathcal{F}_\mathbf{a}(\mathbf{u})$. From Lemma 3, $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathcal{H}_\mathbf{a}$ if and only if $\mathcal{F}_\mathbf{a}(\mathbf{u}) \in [0, s - a_i - 1]$ if and only if $\mathcal{F}_\mathbf{a}(\mathbf{u} + \mathbf{t}) \in [0, s - a_i - 1]$ if and only if $(\mathbf{u} + \mathbf{t}, \mathbf{u} + \mathbf{e}_i + \mathbf{t}) \in \mathcal{H}_\mathbf{a}$. □

We can find generators of the kernel of $\mathcal{F}_\mathbf{a}$ when $d = 3$.

Proposition 2 *If $d = 3$, the kernel of $\mathcal{F}_{\mathbf{a}}$ is*

$$\operatorname{Ker} \mathcal{F}_{\mathbf{a}} = \langle (a_3, 0, -a_1), (0, a_3, -a_2), (a_2, -a_1, 0), (1, 1, 1) \rangle.$$

The result is based on the following well-known lemma.

Lemma 8 *Let K be a subgroup of $\mathbb{Z}^n$ generated by the rows of a $s \times n$ matrix $M \in \mathbb{Z}^{s\times n}$ of rank n. The index $[\mathbb{Z}^n : K]$ is equal to the* gcd *of the n-minors of the matrix M.*

Proof By the rank condition, we have $s \geq n$. Suppose, first, that M is in diagonal form, that is, the diagonal elements of M are $d_1, \ldots, d_n$ and that the other elements are 0; by the rank condition, the d_i are all nonzero. Then the subgroup is $K = d_1\mathbb{Z} \times \cdots \times d_n\mathbb{Z}$, the quotient group is $\mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z}$, and; therefore, the index is $d_1 \ldots d_n$. Moreover, the only nonzero n-minor is $d_1 \ldots d_n$.

In the general case, it is well known that the matrix M may be brought into diagonal form by row and column operations within $\mathbb{Z}^{s\times n}$; moreover, these operations do not change the subgroup, up to change of basis in $\mathbb{Z}^n$; and finally, the gcd of the n-minors is invariant under these operations. Thus the general case follows from the diagonal case. □

Proof of the Proposition The $\supseteq$ part. The kernel of $\mathcal{F}_{\mathbf{a}}$ contains the four vectors, because

$$\begin{aligned}
\mathcal{F}_{\mathbf{a}}(a_3, 0, -a_1) &= a_1a_3 + a_20 + a_3(-a_1) = a_1a_3 - a_3a_1 = 0,\\
\mathcal{F}_{\mathbf{a}}(0, a_3, -a_2) &= a_10 + a_2a_3 + a_3(-a_2) = a_2a_3 - a_3a_2 = 0,\\
\mathcal{F}_{\mathbf{a}}(a_2, -a_1, 0) &= a_1a_2 + a_2(-a_1) + a_30 = a_1a_2 - a_2a_1 = 0.
\end{aligned}$$

and

$$\mathcal{F}_{\mathbf{a}}(1, 1, 1) = a_1 + a_2 + a_3 = \|\mathbf{a}\|_1 = 0.$$

The $\subseteq$ part. Let $K = \langle (a_3, 0, -a_1), (0, a_3, -a_2), (a_2, -a_1, 0), (1, 1, 1) \rangle$. K is a subgroup of $\mathbb{Z}^3$. By showing that the index $[\mathbb{Z}^3 : K]$ is exactly the size $a_1 + a_2 + a_3$ of the image of $\mathcal{F}_{\mathbf{a}}$, we conclude that $K = \operatorname{Ker} \mathcal{F}_{\mathbf{a}}$. The subgroup K is generated by the lines of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1\\ a_3 & 0 & -a_1\\ 0 & a_3 & -a_2\\ a_2 & -a_1 & 0 \end{pmatrix}.$$

From Lemma 8, the index is equal to the gcd of the four 3-minors of the matrix M:

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \end{pmatrix} = a_3(a_1+a_2+a_3),$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ a_3 & 0 & -a_1 \\ a_2 & -a_1 & 0 \end{pmatrix} = -a_1(a_1+a_2+a_3)$$

$$\det\begin{pmatrix} 1 & 1 & 1 \\ 0 & a_3 & -a_2 \\ a_2 & -a_1 & 0 \end{pmatrix} = -a_2(a_1+a_2+a_3),\ \det\begin{pmatrix} a_3 & 0 & -a_1 \\ 0 & a_3 & -a_2 \\ a_2 & -a_1 & 0 \end{pmatrix} = 0.$$

Since a_1, a_2, and a_3 are relatively prime by hypothesis, the index is

$$\begin{aligned}[\mathbb{Z}^3 : K] &= \gcd(a_3(a_1+a_2+a_3), -a_1(a_1+a_2+a_3), -a_2(a_1+a_2+a_3), 0) \\ &= a_1+a_2+a_3.\end{aligned}$$

□

Corollary 1 *The kernel of $\mathcal{F}'_{\mathbf{a}}$ is spanned by the vectors $a_3h_1 - a_1h_3$, $a_3h_2 - a_2h_3$, $a_2h_1 - a_1h_2$.*

Proof This is because $\pi(\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}) = \mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$. Indeed, $\mathcal{F}_{\mathbf{a}} = \mathcal{F}'_{\mathbf{a}} \circ \pi$ and $\pi(1,1,1) = 0$. □

It is likely that Proposition 2 and Corollary 1 have an evident extension to any dimension. The proof requires a higher minor calculation. We leave this to the interested reader.

3.4 The Graph $\mathcal{G}_{\mathbf{a}}$

Let $d \geq 2$ be an integer and $\mathbf{a}$ as before. The graph $\mathcal{G}_{\mathbf{a}}$ of normal vector $\mathbf{a} \in \mathbb{Z}^d$ is the directed graph $\mathcal{G}_{\mathbf{a}} = \mathcal{F}_{\mathbf{a}}(\mathcal{H}_{\mathbf{a}})$. It is also equal to

$$\mathcal{G}_{\mathbf{a}} = \{(k, k+a_i) \mid k \in \mathbb{Z}/s\mathbb{Z}, 1 \leq i \leq d \text{ and } k < k+a_i\}.$$

Two examples are shown in Fig. 4. Since the graph $\mathcal{G}_{\mathbf{a}}$ is isomorphic to the quotients $\mathcal{H}_{\mathbf{a}}/\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}$ (and also $I_{\mathbf{a}}/\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$), we call it *Christoffel graph* as well, because $\mathcal{G}_{\mathbf{a}}$ is the part of $\mathcal{H}_{\mathbf{a}}$ in its fundamental domain. The graph $\mathcal{G}_{\mathbf{a}}$ can be embedded in a torus. Indeed, the graph $I_{\mathbf{a}}$ lives in the diagonal plane $\mathcal{D} \simeq \mathbb{R}^{d-1}$ and is invariant under the group $\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$. The quotient $\mathcal{D}/\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$ is a torus and contains the graph $I_{\mathbf{a}}/\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}' \simeq \mathcal{G}_{\mathbf{a}}$ (see Fig. 5).

The vertices of the Christoffel graph $\mathcal{H}_{\mathbf{a}}$ and their image under the function $\mathcal{F}_{\mathbf{a}}$ correspond to what is called *roundwalk* in [7]. Their contribution allows us to construct larger and larger domains of roundwalks by iteration of extension rules.

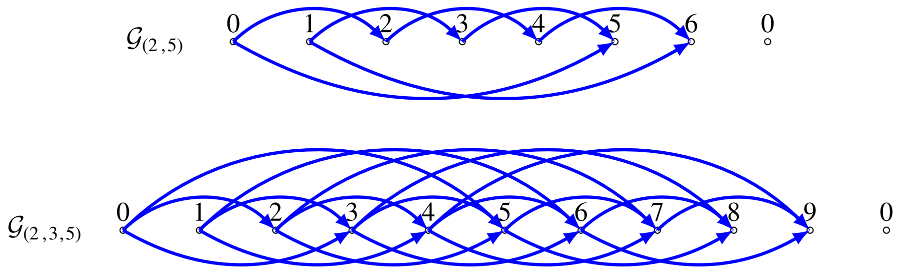


Fig. 4 The Christoffel graphs $\mathcal{G}_{\mathbf{a}}$ for $\mathbf{a} = (2, 5)$ and $\mathbf{a} = (2, 3, 5)$

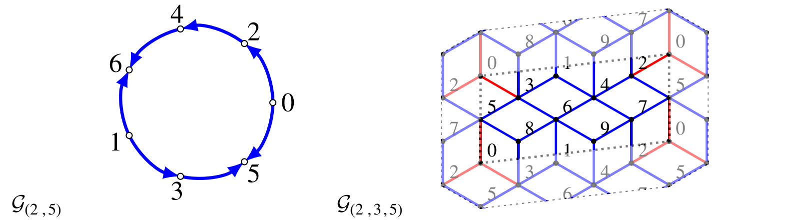


Fig. 5 The Christoffel graphs $\mathcal{G}_{\mathbf{a}}$ for $\mathbf{a} = (2, 5)$ and $\mathbf{a} = (2, 3, 5)$ can be embedded in the torus $\mathcal{D}/\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$

The Christoffel graph has a natural representation inside $I_{\mathbf{a}}$. We define this for $d = 3$, leaving the generalizations for elsewhere. Recall that the lattice L, defined in Sect. 3, is a free abelian group of rank 2, spanned by the 3 vectors $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ with $\mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3 = 0$. Moreover, the homomorphism $\mathcal{F}'_{\mathbf{a}} : L \to \mathbb{Z}/s\mathbb{Z}$ maps $\mathbf{h}_i$ onto a_i, with $s = a_1 + a_2 + a_3$, and the a_i are relatively prime; therefore, the mapping is surjective.

Choose some parallelogram in the plane $\mathcal{D}$, which is a fundamental domain for its discrete subgroup $\mathrm{Ker}\,\mathcal{F}_{\mathbf{a}}{}'$. We may assume that O is a vertex of this parallelogram. Then $\mathcal{F}'_{\mathbf{a}}$ induces a bijection between $\mathbb{Z}/s\mathbb{Z}$ and the integral points inside the parallelogram, excluding those lying on the two edges not containing O. It is such a parallelogram, with the part of the edges of $I_{\mathbf{a}}$ which lie inside it, that we may call a *Christoffel parallelogram*. We may consider this as the generalization in dimension 3 of Christoffel words. Such a parallelogram tiles the plane $\mathcal{D}$ and completely codes the graph $I_{\mathbf{a}}$. Furthermore it is in bijection with the Christoffel graph, as is easily verified. Examples are seen in Fig. 6.

Given a Christoffel parallelogram, we may follow its edges and we obtain a $\mathbb{Z}$-linear combination of $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$. Replacing $\mathbf{h}_i$ by $\mathbf{e}_i$, we obtain a 3-dimensional vector. It is well-defined and does not depend on the chosen path in the parallelogram, only on the first and last vertices; this follows since for each rhombus in the parallelogram, the closed path around it has label $\mathbf{h}_i + \mathbf{h}_j - \mathbf{h}_i - \mathbf{h}_j = 0$. Applying this construction to two adjacent sides of a parallelogram, for example, in Figs. 3 (right) and 5 (right), we obtain the paths $\mathbf{h}_2 + \mathbf{h}_1 - \mathbf{h}_3$ and $\mathbf{h}_2 - \mathbf{h}_1 + \mathbf{h}_2 - \mathbf{h}_1 - \mathbf{h}_1$; hence, the two three-dimensional vectors $\mathbf{u} = \mathbf{e}_2 + \mathbf{e}_1 - \mathbf{e}_3$ and $\mathbf{v} = -3\mathbf{e}_1 + 2\mathbf{e}_2$. This defines the matrix

$$\begin{pmatrix} 1 & 1 & -1 \\ -3 & 2 & 0 \end{pmatrix}$$

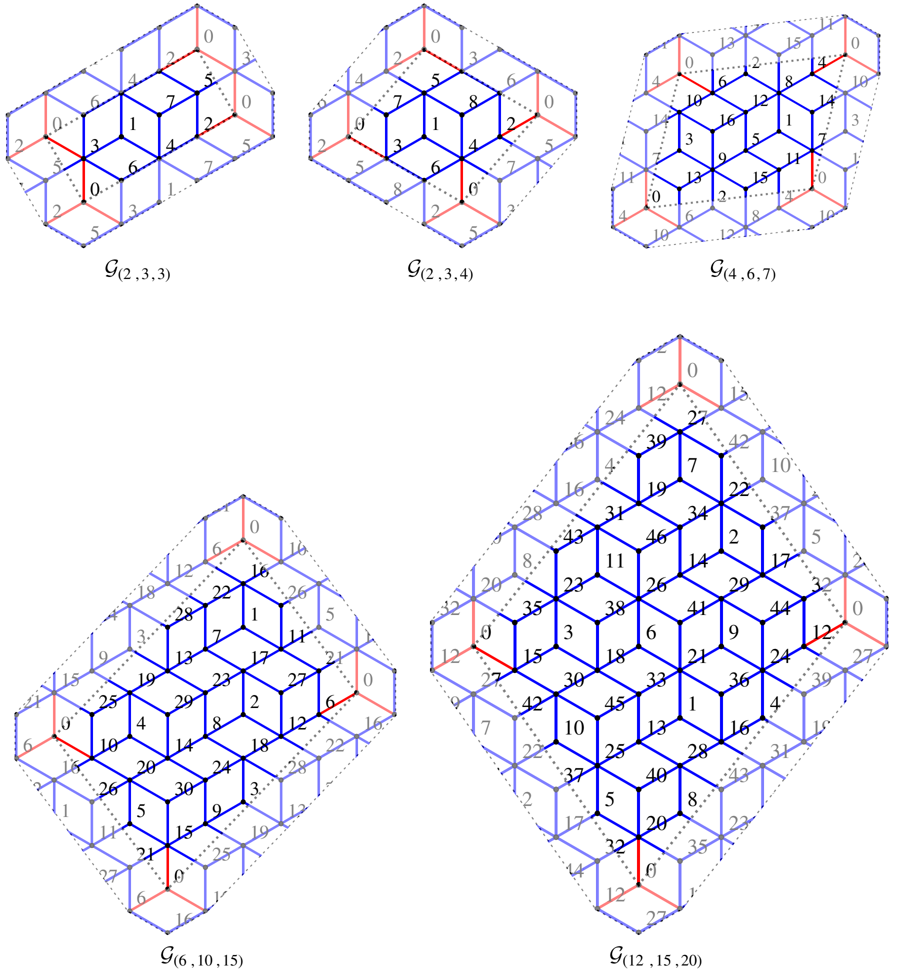


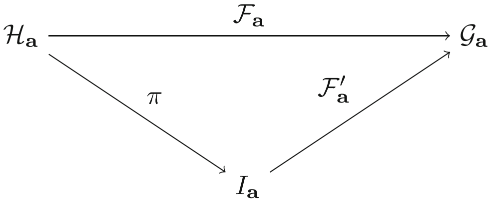
Fig. 6 Some Christoffel graphs in dimension $d = 3$. Legs (defined in Sect. 4) are the edges of the Christoffel graphs incident to zero. All other edges constitute the body of the Christoffel graph

from which we deduce the value of the vector product (or cross product) by taking the 2-minors $\mathbf{u} \wedge \mathbf{v} = (2, 3, 5)$. We recover the vector $\mathbf{a} = \mathbf{u} \wedge \mathbf{v}$. This is true in general.

Proposition 3 *Let* $u, \mathbf{v} \in \mathbb{Z}^3$ *be two adjacent sides of a Christoffel parallelogram of normal vector* $\mathbf{a}$. *Then* $\mathbf{u} \wedge \mathbf{v} = \pm \mathbf{a}$.

Proof It is well known that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{u} \wedge \mathbf{v}$. Hence, it is enough to prove that the coordinates of $\mathbf{u} \wedge \mathbf{v}$ are relatively prime. This follows since, by construction, the 3-dimensional parallelogram constructed on $\mathbf{u}$, $\mathbf{v}$ contains no integer points, except the 4 vertices; therefore, the matrix with rows $\mathbf{u}$, $\mathbf{v}$ is such that the gcd of the i-minors is 1, for $i = 1$ and 2. □

Remark 1 The graphs $\mathcal{H}_{\mathbf{a}}, I_{\mathbf{a}}, \mathcal{G}_{\mathbf{a}}$ are compatible, in the sense that $I_{\mathbf{a}}$ is the image under π of $\mathcal{H}_{\mathbf{a}}$, and $\mathcal{G}_{\mathbf{a}}$ is the image under $\mathcal{F}_{\mathbf{a}}$ of $\mathcal{H}_{\mathbf{a}}$ and also the image of $I_{\mathbf{a}}$ under $\mathcal{F}'_{\mathbf{a}}$.



3.5 The Graph $\mathcal{H}_{\mathbf{a},\omega}$

In this section, we extend the definition of Christoffel graphs to the digital plane such that the width ω is smaller than $s = \|\mathbf{a}\|_1 = \sum a_i$, where $\mathbf{a} \in \mathbb{N}^d$ is a vector of relatively prime positive integers as before. We consider only width ω such that s/ω is a positive integer strictly smaller than the dimension d: $0 < s/\omega < d$. We define the mapping $\mathcal{F}_{\mathbf{a},\omega} : \mathbb{Z}^d \to \mathbb{Z}/\omega\mathbb{Z}$ sending each integral vector $(x_1, \ldots, x_d)$ onto $\sum_i a_i x_i \bmod \omega$. We identify $\mathbb{Z}/\omega\mathbb{Z}$ and $\{0, 1, \ldots, \omega - 1\}$. A total order on $\mathbb{Z}/\omega\mathbb{Z}$ is defined correspondingly. The *Christoffel graph of normal vector* $\mathbf{a} \in \mathbb{N}^d$ *of width* ω is the subset of edges $\mathcal{H}_{\mathbf{a},\omega} \subseteq \mathbb{E}_d$ defined by

$$\mathcal{H}_{\mathbf{a},\omega} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) < \mathcal{F}_{\mathbf{a},\omega}(\mathbf{v})\}.$$

This graph is related but does not correspond exactly to the digital plane of width ω. In fact, $\mathcal{H}_{\mathbf{a},\omega}$ can be obtained by the superposition of the s/ω digital plane of width ω. The definition of $\mathcal{H}_{\mathbf{a},\omega}$ is motivated by Pirillo's theorem, because this is what allows us to generalize Pirillo's theorem in an arbitrary dimension (see Theorem 3). Of course, if $\omega = s$, then $\mathcal{H}_{\mathbf{a},\omega} = \mathcal{H}_{\mathbf{a}}$ is the Christoffel graph of normal vector $\mathbf{a}$. Also, if $d = 2$, then $s = \omega$. If $d = 3$, then either $\omega = s$ or $\omega = s/2$. If $d = 4$, then either $\omega = s$, $\omega = s/2$ or $\omega = s/3$ and so on for $d \geq 5$. If s is a prime number, then $\omega = s$.

As earlier, we define the projected graphs $\mathcal{G}_{\mathbf{a},\omega} := \mathcal{F}_{\mathbf{a},\omega}(\mathcal{H}_{\mathbf{a},\omega})$ and $I_{\mathbf{a},\omega} := \pi(\mathcal{H}_{\mathbf{a},\omega})$. The Christoffel graph $\mathcal{G}_{\mathbf{a},\omega}$ for the vector $\mathbf{a} = (15, 11, 10)$ of width $\omega = s = 36$ is shown in Fig. 7 (left). The Christoffel graph $\mathcal{G}_{\mathbf{a},\omega}$ for the vector $\mathbf{a} = (15, 11, 10)$ of width $\omega = 18 = s/2$ is shown in Fig. 7 (right) and a larger part is shown in Fig. 8.

The next lemma gives an equivalent definition of the edges of the graph $\mathcal{H}_{\mathbf{a},\omega}$.

Lemma 9 *Let* $(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d$ *such that* $\mathbf{v} - \mathbf{u} = \mathbf{e}_i$ *for some* $1 \leq i \leq d$. *Then,*

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a},\omega} &\iff \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) \in [0, \omega - a_i - 1] \\ &\iff \mathcal{F}_{\mathbf{a},\omega}(\mathbf{v}) \in [a_i, \omega - 1], \end{aligned} \tag{4}$$

$$\begin{aligned} (\mathbf{u}, \mathbf{v}) \notin \mathcal{H}_{\mathbf{a},\omega} &\iff \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) \in [\omega - a_i, \omega - 1] \\ &\iff \mathcal{F}_{\mathbf{a},\omega}(\mathbf{v}) \in [0, a_i - 1]. \end{aligned} \tag{5}$$

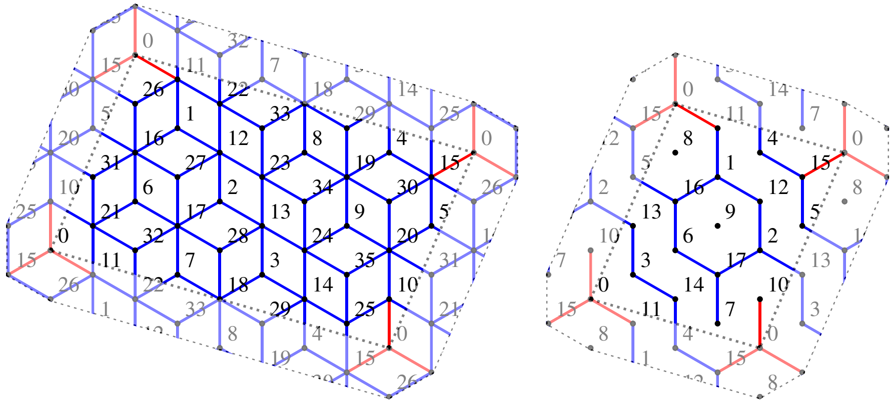


Fig. 7 *Left* The Christoffel graph $\mathcal{G}_{\mathbf{a}}$ for the vector $\mathbf{a} = (15, 11, 10)$. *Right* The Christoffel graph $\mathcal{G}_{\mathbf{a},\omega}$ of width $\omega = 18$ for the vector $\mathbf{a} = (15, 11, 10)$

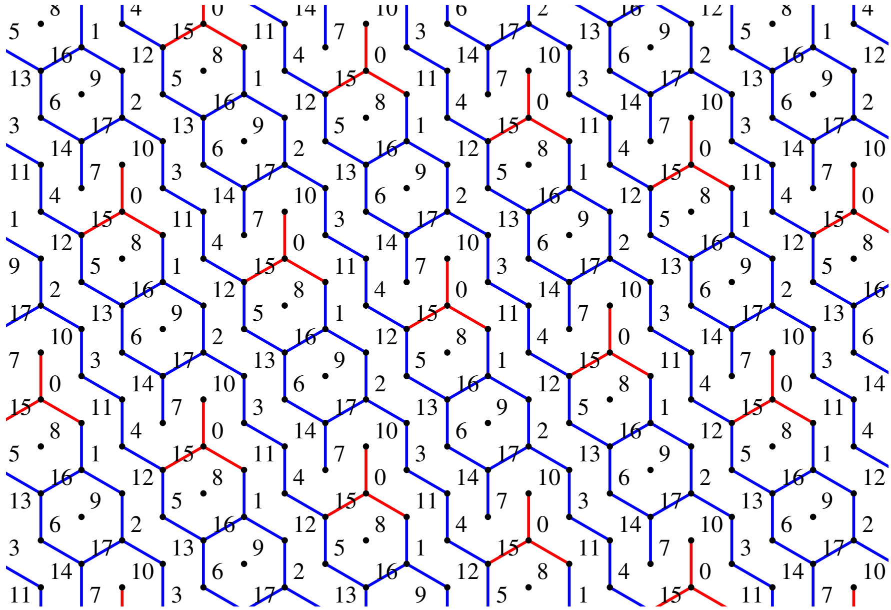


Fig. 8 The Christoffel graph $I_{\mathbf{a},\omega}$ of width $\omega = 18$ for the vector $\mathbf{a} = (15, 11, 10)$. It corresponds to the union of two digital planes of width ω

4 Flip, Reversal, and Translation

In this section, we define the flip, reversal, and translation of a set of edges. We define the operations for a set of edges $X \subseteq \mathbb{E}_d$, but they extend naturally to a set of edges of the forms $\pi(X)$ and $\mathcal{F}_{\mathbf{a}}(X)$ (see Definition 6 below). In order to define the flip operation, we need to define the edges incident to zero.

Definition 2 (*Edges of* $\mathbb{E}_d$ *incident to zero*) Let $d \geq 2$ be an integer and $\mathbf{a} \in \mathbb{Z}^d$ be a vector of relatively prime positive integers. The set of *edges of* $\mathbb{E}_d$ *incident to zero* is

$$\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{u}) = 0 \text{ or } \mathcal{F}_{\mathbf{a}}(\mathbf{v}) = 0\}.$$

Definition 3 (*Body, Legs*) Let $X \subseteq \mathbb{E}_d$. The set $X \setminus \mathcal{Q}$ is the *body* and the edges of $X \cap \mathcal{Q}$ are the *legs* of X.

See Fig. 6 where the legs of graphs $\mathcal{G}_{\mathbf{a}}$ are represented in red and the body in blue. The FLIP is an operation which generalizes the function $amb \mapsto bma$ defined for Christoffel words. While we define the flip on graphs, it can also be seen as a flip in a rhombus tiling when $d = 3$ [4,9,10].

Definition 4 (*Flip*) For a subset of edges $X \subseteq \mathbb{E}_d$, we define the FLIP operation, which exchanges edges incident to zero:

$$\text{FLIP} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

We see that $\text{FLIP}(X)$ exchanges the legs of X and keeps the body of X invariant. If $(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d$, then the reversal edge $(-\mathbf{v}, -\mathbf{u}) \in \mathbb{E}_d$ is also an edge of the hypercubic lattice and similarly for the translated edge $(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \in \mathbb{E}_d$ for all $\mathbf{t} \in \mathbb{Z}^d$. The reversal and translation operations extend on subsets of edges as follows:

Definition 5 (*Reversal, Translation*) Let $X \subseteq \mathbb{E}_d$ be a subset of edges. We define the *reversal* $-X$ of X and the *translation* $X + \mathbf{t}$, for some $\mathbf{t} \in \mathbb{Z}^d$, of X as

$$-X = \{(-\mathbf{v}, -\mathbf{u}) \mid (\mathbf{u}, \mathbf{v}) \in X\} \quad \text{and} \quad X + \mathbf{t} = \{(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \mid (\mathbf{u}, \mathbf{v}) \in X\}.$$

Definition 6 (*Flip, Reversal, Translation*) Let $X \subseteq \mathbb{E}_d$. The flip, reversal, and translation of set of edges of the forms $\pi(X)$ and $\mathcal{F}_{\mathbf{a}}(X)$ are defined naturally by commutativity:

$$\begin{aligned}
&\text{FLIP}(\pi(X)) := \pi(\text{FLIP}(X)), \quad -(\pi(X)) := \pi(-X), \quad \pi(X) + \pi(\mathbf{t}) := \pi(X + \mathbf{t}),\\
&\text{FLIP}(\mathcal{F}_{\mathbf{a}}(X)) := \mathcal{F}_{\mathbf{a}}(\text{FLIP}(X)), \; -(\mathcal{F}_{\mathbf{a}}(X)) := \mathcal{F}_{\mathbf{a}}(-X), \; \mathcal{F}_{\mathbf{a}}(X) + \mathcal{F}_{\mathbf{a}}(\mathbf{t}) := \mathcal{F}_{\mathbf{a}}(X + \mathbf{t}).
\end{aligned}$$

Therefore, statements proven for $\mathcal{H}_{\mathbf{a}}$ using flip, reversal, and translated operations are also true for $I_{\mathbf{a}}$ and $\mathcal{G}_{\mathbf{a}}$. For example, the goal of the next section is to show that $\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$ for some $\mathbf{t} \in \mathbb{Z}^d$. If such an equation is satisfied for $\mathcal{H}_{\mathbf{a}}$, it is clear from Definition 6 that $I_{\mathbf{a}} + \pi(\mathbf{t}) = \text{FLIP}(I_{\mathbf{a}})$ and $\mathcal{G}_{\mathbf{a}} + \mathcal{F}_{\mathbf{a}}(\mathbf{t}) = \text{FLIP}(\mathcal{G}_{\mathbf{a}})$.

5 Flipping is Translating

In this section, we show that the flip of the Christoffel graph $\mathcal{H}_{\mathbf{a}}$ is a translation of $\mathcal{H}_{\mathbf{a}}$; this is a generalization of one implication of Theorem 1. We also show that the body of $\mathcal{H}_{\mathbf{a}}$ is symmetric and, as a consequence, we obtain that a Christoffel graph is a translation of its reversal. The results stated in this section are stated and proved for $\mathcal{H}_{\mathbf{a}}$, but they are valid for $I_{\mathbf{a}}$ and $\mathcal{G}_{\mathbf{a}}$ by Definition 6.

The following lemma describes the legs of $\mathcal{H}_{\mathbf{a}}$.

Lemma 10 (Legs of $\mathcal{H}_{\mathbf{a}}$) *An edge* $(\mathbf{u}, \mathbf{v})$ *is a leg of* $\mathcal{H}_{\mathbf{a}}$ *if and only if* $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) = 0$.

Proof We have $(-\mathbf{e}_i, \mathbf{0}) \notin \mathcal{H}_{\mathbf{a}}$ because there is no $\mathbf{u} \in \mathbb{Z}^d$ such that $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) < \mathcal{F}_{\mathbf{a}}(\mathbf{0}) = 0$. Moreover, $(\mathbf{0}, \mathbf{e}_i) \in \mathcal{H}_{\mathbf{a}}$ for each i, $1 \le i \le d$, because $\mathcal{F}_{\mathbf{a}}(\mathbf{0}) = 0 < a_i = \mathcal{F}_{\mathbf{a}}(\mathbf{e}_i)$. □

We now show that the body of a Christoffel graph is *symmetric*, i.e., it is equal to its reversal. This generalizes the fact that central words are palindromes.

Lemma 11 *The body of* $\mathcal{H}_{\mathbf{a}}$ *is symmetric, i.e.,* $-(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$.

Proof It is sufficient to prove $-(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) \supseteq \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$, the other inclusion being equivalent, since the symmetry is involutive. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$. Then $\mathbf{v} - \mathbf{u} = \mathbf{e}_i$ for some $1 \le i \le d$. Then $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [0, s - a_i - 1]$ by Lemma 3 and $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \notin \{0, s - a_i\}$ so that $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [1, s - a_i - 1]$. Thus $\mathcal{F}_{\mathbf{a}}(-\mathbf{u}) = s - \mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [a_i + 1, s - 1]$. We obtain that $(-\mathbf{v}, -\mathbf{u}) \in \mathcal{H}_{\mathbf{a}}$ by Lemma 3 because $-\mathbf{u} - (-\mathbf{v}) = \mathbf{v} - \mathbf{u} = \mathbf{e}_i$. Since $\mathcal{Q} = -\mathcal{Q}$, $(\mathbf{u}, \mathbf{v}) \notin \mathcal{Q}$ implies that $(-\mathbf{v}, -\mathbf{u}) \notin \mathcal{Q}$. We conclude $(-\mathbf{v}, -\mathbf{u}) \in \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$ and $(\mathbf{u}, \mathbf{v}) \in -(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q})$. □

We now show that the reversal is equal to the flip of a Christoffel graph. This generalizes the fact that the reversal $\widetilde{amb}$ of a Christoffel word is equal to bma.

Lemma 12 *The reversal of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,* $-\mathcal{H}_{\mathbf{a}} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$.

Proof For $\mathcal{H}_{\mathbf{a}}$, we have to show that $-\mathcal{H}_{\mathbf{a}} = (\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus \mathcal{H}_{\mathbf{a}})$. We prove the result in two parts since $-\mathcal{H}_{\mathbf{a}} = \big((-\mathcal{H}_{\mathbf{a}}) \setminus \mathcal{Q}\big) \cup \big((-\mathcal{H}_{\mathbf{a}}) \cap \mathcal{Q}\big)$ is the disjoint union of a part outside of $\mathcal{Q}$ and a part inside of $\mathcal{Q}$. Outside of $\mathcal{Q}$: since $\mathcal{Q}$ is symmetric and because $\mathcal{H}_{\mathbf{a}}$ is symmetric from Lemma 11, we have $-(\mathcal{H}_{\mathbf{a}}) \setminus \mathcal{Q} = -(\mathcal{H}_{\mathbf{a}}) \setminus -\mathcal{Q} = -(\mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}) = \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$. Inside of $\mathcal{Q}$: we have $(-\mathcal{H}_{\mathbf{a}}) \cap \mathcal{Q} = -(\mathcal{H}_{\mathbf{a}} \cap \mathcal{Q}) = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d : \mathcal{F}_{\mathbf{a}}(\mathbf{v}) = 0\} = \mathcal{Q} \setminus \mathcal{H}_{\mathbf{a}}$ by Lemma 10. □

We now show that the flip of a Christoffel graph $\mathcal{H}_{\mathbf{a}}$ is equal to a translation of $\mathcal{H}_{\mathbf{a}}$. It generalizes one implication of Theorem 1. It corresponds to the fact that a Christoffel word amb is conjugate to bma. Proposition 4 is illustrated in Figs. 9 and 10.

Proposition 4 *Let* $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$. *The translation by* $\mathbf{t}$ *of* $\mathcal{H}_{\mathbf{a}}$ *is equal to its flip, i.e.,* $\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a}})$.

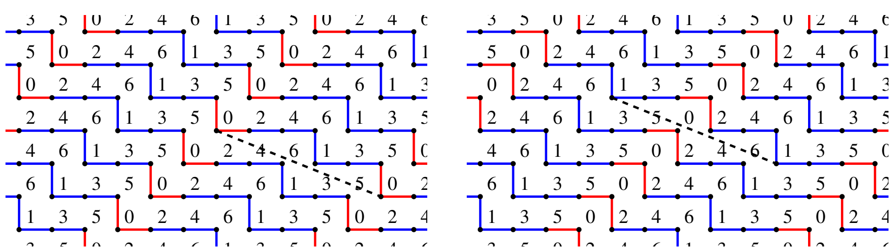


Fig. 9 *Left* The graph $\mathcal{H}_{\mathbf{a}}$ with $\mathbf{a} = (2, 5)$. *Right* FLIP($\mathcal{H}_{\mathbf{a}}$)

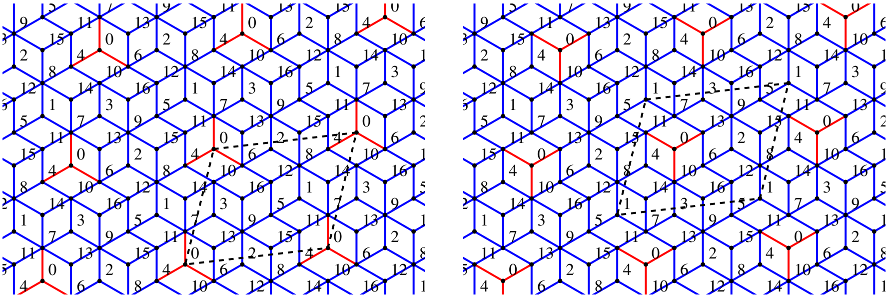


Fig. 10 *Left* The graph $I_{\mathbf{a}}$ with $\mathbf{a} = (4, 6, 7)$. *Right* FLIP($I_{\mathbf{a}}$). Consider the Christoffel parallelogram P with vertices labeled by 0 embedded in $I_{\mathbf{a}}$. The parallelogram P also appears in the graph FLIP($I_{\mathbf{a}}$) with vertices labeled by 1

Note that, since $\mathcal{F}_{\mathbf{a}}$ is surjective, there exists indeed $\mathbf{t} \in \mathbb{Z}^d$ such that $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$. Also, $\mathcal{F}_{\mathbf{a}}(-\mathbf{t}) = -\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = -1$.

Proof We prove the result in two parts since $\mathcal{H}_{\mathbf{a}} + \mathbf{t}$ is the disjoint union of a part outside of $\mathcal{Q}$ and a part inside of $\mathcal{Q}$:

$$\mathcal{H}_{\mathbf{a}} + \mathbf{t} = \big((\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \setminus \mathcal{Q}\big) \cup \big((\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \cap \mathcal{Q}\big).$$

1. $(\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \setminus \mathcal{Q} \supseteq \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$. Suppose that $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$. Thus, $\mathbf{v} - \mathbf{u} = \mathbf{e}_i$ for some $1 \le i \le d$. Then, $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [0, s - a_i - 1]$ by Lemma 3 and, since the edge is not a leg, $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \notin \{0, s - a_i\}$. Hence, $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [1, s - a_i - 1]$. Then $\mathcal{F}_{\mathbf{a}}(\mathbf{u} - \mathbf{t}) = \mathcal{F}_{\mathbf{a}}(\mathbf{u}) - 1 \in [0, s - a_i - 2]$, which implies that $(\mathbf{u} - \mathbf{t}, \mathbf{v} - \mathbf{t}) \in \mathcal{H}_{\mathbf{a}}$. Then $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}} + \mathbf{t}$.
2. $(\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \setminus \mathcal{Q} \subseteq \mathcal{H}_{\mathbf{a}} \setminus \mathcal{Q}$. Let $(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \in (\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \setminus \mathcal{Q}$ for some edge $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}}$. Then, $\mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{t}) \notin \{0, s - a_i\}$. From Lemma 3, $\mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{t}) = \mathcal{F}_{\mathbf{a}}(\mathbf{u}) + 1 \in [1, s - a_i]$. Therefore, $\mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{t}) \in [1, s - a_i - 1]$, and we conclude that $(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \in \mathcal{H}_{\mathbf{a}}$.
3. $(\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \cap \mathcal{Q} \supseteq \mathcal{Q} \setminus \mathcal{H}_{\mathbf{a}}$. Let $(\mathbf{u}, \mathbf{v}) \in \mathcal{Q} \setminus \mathcal{H}_{\mathbf{a}}$. Then, $\mathcal{F}_{\mathbf{a}}(\mathbf{v}) = 0$, which implies that $\mathcal{F}_{\mathbf{a}}(\mathbf{v} - \mathbf{t}) = s - 1$. By Lemma 3, we have $(\mathbf{u} - \mathbf{t}, \mathbf{v} - \mathbf{t}) \in \mathcal{H}_{\mathbf{a}}$ so that $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}} + \mathbf{t}$.
4. $(\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \cap \mathcal{Q} \subseteq \mathcal{Q} \setminus \mathcal{H}_{\mathbf{a}}$. Let $(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \in (\mathcal{H}_{\mathbf{a}} + \mathbf{t}) \cap \mathcal{Q}$ for some edge $(\mathbf{u}, \mathbf{v}) \in \mathcal{H}_{\mathbf{a}}$. Either $\mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{t}) = 0$ or $\mathcal{F}_{\mathbf{a}}(\mathbf{v} + \mathbf{t}) = 0$. If $\mathcal{F}_{\mathbf{a}}(\mathbf{u} + \mathbf{t}) = 0$, then $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) = s - 1$, which implies that $(\mathbf{u}, \mathbf{v}) \notin \mathcal{H}_{\mathbf{a}}$ by Lemma 3, a contradiction. One must have $\mathcal{F}_{\mathbf{a}}(\mathbf{v} + \mathbf{t}) = 0$, which implies that $(\mathbf{u} + \mathbf{t}, \mathbf{v} + \mathbf{t}) \notin \mathcal{H}_{\mathbf{a}}$. □

The previous proposition proves that the body of a Christoffel graph $\mathcal{H}_{\mathbf{a}}$ has a period. This generalizes the fact that central words of length $p + q - 2$ have periods p and q (remark that p and $-q$ is the same period mod the length of the Christoffel words $|w| = p + q$). Indeed, let P be some parallelogram and M be an inner point. Consider the 4 vectors with origin equal to one of the vertices of P and with end M. Then, for each point X in P, there is one of these vectors, $\mathbf{v}$ say, such that the segment $[X, X + \mathbf{v}]$ is contained in P. We leave the verification of this to the reader. It follows that a Christoffel parallelogram may be reconstructed from the edges incident to zero by applying translations which stay completely in the parallelogram. This is completely analogous to the fact that a central word is completely determined by its two periods.

The next result generalizes the fact that the reversal $\widetilde{w}$ of a Christoffel word w is conjugate to w. This is not a characteristic property of Christoffel words, because it is satisfied for all words that are the product of two palindromes.

Corollary 2 *Let* $\mathbf{t} \in \mathbb{Z}^d$ *be such that* $\mathcal{F}_{\mathbf{a}}(\mathbf{t}) = 1$. *Then* $-\mathcal{H}_{\mathbf{a}} = \mathcal{H}_{\mathbf{a}} + \mathbf{t}$.

Proof Follows from Lemma 12 and Proposition 4. □

Corollary 3 (i) *The body of a Christoffel parallelogram is symmetric with respect to its center.*
(ii) *Consider a Christoffel parallelogram* P *embedded in* $I_{\mathbf{a}}$. *The parallelogram obtained by symmetry of* P *with respect to its center appears as a translation of* P *within* $I_{\mathbf{a}}$, *and is also equal to the flip of* P.

We, thus, have obtained a generalization of (i) that a central word is a palindrome; (ii) the reversal of a Christoffel word amb is conjugate to it, and equal to bma. The corollary can be checked in Figs. 6, 9 and 10

6 Higher-Dimensional Pirillo's Theorem

In this section, we study the converse of Proposition 4. In other words, does the fact of being a translation of its flip a characteristic property of Christoffel graphs as it is the case for Christoffel words? We show that it must be a Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ for some vector $\mathbf{a} \in \mathbb{Z}^d$ and width ω. For $d = 3$, we show that parallelograms that are translated to their flip are Christoffel parallelograms or their edge-complement.

Let K be a subgroup of $\mathbb{Z}^d$ for some integer $d \geq 2$ such that the index $[\mathbb{Z}^d : K]$ is finite and $\sum_{i=1}^{d} \mathbf{e}_i = (1, 1, \ldots, 1) \in K$. Let $\mathcal{Q}$ be the set of *edges of* $\mathbb{E}_d$ *incident to zero mod* K:

$$\mathcal{Q} = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ or } \mathbf{v} \in K\}.$$

For a subset of edges $X \subseteq \mathbb{E}_d$, we redefine the FLIP operation according to the above set $\mathcal{Q}$:

$$\text{FLIP} : X \mapsto (X \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus X).$$

In what follows, we assume that $M \subseteq \mathbb{E}_d$ is a set of edges such that

- M is invariant for the group of translations K;
- $\text{FLIP}(M) = M + \mathbf{t}$ for some $\mathbf{t} \in \mathbb{Z}^d$.

If $\text{FLIP}(M) = M + \mathbf{t}$, then for each i, $(\mathbf{0}, \mathbf{e}_i) \in M$ or $(-\mathbf{e}_i, \mathbf{0}) \in M$ but not both. Otherwise the number of edges parallel to the vector $\mathbf{e}_i$ is not preserved by the FLIP and the equation cannot be satisfied. For each integer i with $1 \leq i \leq d$, we suppose that $(\mathbf{0}, \mathbf{e}_i) \in M$ and $(-\mathbf{e}_i, \mathbf{0}) \notin M$ like it is the case for Christoffel graphs. In other words, the legs of M are

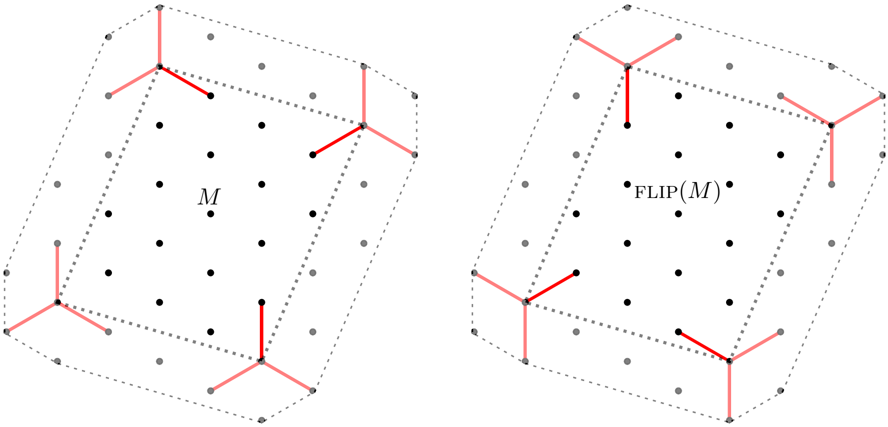


Fig. 11 *Left* M. *Right* FLIP(M) for the subgroup $K = \langle(0, 4, 1), (-2, 0, 3), (1, 1, 1)\rangle$

– $\mathcal{Q} \cap M = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d \mid \mathbf{u} \in K \text{ and } 1 \le i \le d\}$.

We consider the following question in this section: for which set of edges $M \subseteq \mathbb{E}_d$ satisfying the above three conditions, does there exist a translation $\mathbf{t} \in \mathbb{Z}^d$ such that FLIP$(M) = M + \mathbf{t}$ (see Fig. 11)?

Already, we may show that $\mathbf{t} \notin K$ from the first two hypotheses. Thus, $\mathbf{t}$ has a positive and finite order in the group $\mathbb{Z}^d/K$.

Lemma 13 *We have* $\mathbf{t} \in \mathbb{Z}^d \setminus K$.

Proof Suppose that $\mathbf{t} \in K$. Since $M \subseteq \mathbb{E}_d$ is invariant for the group of translations K, we have $M + \mathbf{t} = M$. Therefore, the equation FLIP$(M) = M + \mathbf{t} = M$ holds. This is a contradiction since the function FLIP has no fixed point. Indeed, consider some edge $h \in \mathcal{Q}$. If $h \in M$, then $h \notin$ FLIP$(M) = M$ which is a contradiction. On the other hand, if $h \notin M$, then $h \in$ FLIP$(M) = M$ which is a contradiction. □

The next lemma is technical but useful for the proposition that follows.

Lemma 14 *Let* $\mathbf{t} \in \mathbb{Z}^d$ *be a translation. Let* $X \subseteq \mathbb{E}_d$ *and* $h \in \mathbb{E}_d$. *We have*

(i) *If* $h \in X$, *then* $h + \mathbf{t} \notin \mathcal{Q}$ *if and only if* $h + \mathbf{t} \in$ FLIP$(X + \mathbf{t})$.
(ii) *If* $h \notin X$, *then* $h + \mathbf{t} \in \mathcal{Q}$ *if and only if* $h + \mathbf{t} \in$ FLIP$(X + \mathbf{t})$.

Proof We have FLIP$(X + \mathbf{t}) = ((X + \mathbf{t}) \setminus \mathcal{Q}) \cup (\mathcal{Q} \setminus (X + \mathbf{t}))$.

(i) If $h \in X$, then $h + \mathbf{t} \in X + \mathbf{t}$. If $h + \mathbf{t} \notin \mathcal{Q}$, then $h + \mathbf{t} \in (X + \mathbf{t}) \setminus \mathcal{Q} \subseteq$ FLIP$(X + \mathbf{t})$. If $h + \mathbf{t} \in \mathcal{Q}$, then $h + \mathbf{t} \in (X + \mathbf{t}) \cap \mathcal{Q}$. Therefore, $h + \mathbf{t} \notin$ FLIP$(X + \mathbf{t})$.
(ii) If $h \notin X$ then $h + \mathbf{t} \notin X + \mathbf{t}$. If $h + \mathbf{t} \in \mathcal{Q}$, then $h + \mathbf{t} \in \mathcal{Q} \setminus (X + \mathbf{t}) \subseteq$ FLIP$(X + \mathbf{t})$. If $h + \mathbf{t} \notin \mathcal{Q}$, then $h + \mathbf{t} \notin (X + \mathbf{t}) \cup \mathcal{Q} \supseteq$ FLIP$(X + \mathbf{t})$. Therefore, $h + \mathbf{t} \notin$ FLIP$(X + \mathbf{t})$.

□

Proposition 5 *For all* i, *with* $1 \le i \le d$, *there exists a unique integer* b_i, $0 < b_i < \omega$, *such that* $\mathbf{e}_i + b_i\mathbf{t} \in K$ *where* ω *is the order of* $\mathbf{t}$ *in the group* $\mathbb{Z}^d/K$. *Moreover,*

$$(\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} \begin{cases} \in M & \text{if } 0 \le k < b_i, \\ \notin M & \text{if } b_i \le k < \omega. \end{cases}$$

In the following proof, for two elements $\mathbf{u}, \mathbf{u}' \in \mathbb{Z}^d$ the notation $\mathbf{u} \equiv \mathbf{u}'$ is used when $\mathbf{u}' - \mathbf{u} \in K$. The notation is also used for two edges $(\mathbf{u}, \mathbf{v}), (\mathbf{u}', \mathbf{v}') \in \mathbb{E}_d$: $(\mathbf{u}, \mathbf{v}) \equiv (\mathbf{u}', \mathbf{v}')$ if and only if $\mathbf{u}' - \mathbf{u} = \mathbf{v}' - \mathbf{v} \in K$. Finally, we denote by $\mathbf{0}$ the zero of $\mathbb{Z}^d/K$.

Proof Let $\omega = \min\{k > 0 \mid k\mathbf{t} \in K\}$ be the order of $\mathbf{t}$ in $\mathbb{Z}^d/K$. Consider the orbit under the translation $\mathbf{t}$ of the edge $h = (\mathbf{0}, \mathbf{e}_i) \in \mathcal{Q} \cap M$ for some i such that $1 \leq i \leq d$. We have that $h + \omega\mathbf{t} \equiv h \in \mathcal{Q} \cap M$. We want to show that there exists b_i such that $0 < b_i < \omega$ and $h + b_i t \equiv (-\mathbf{e}_i, \mathbf{0}) \in \mathcal{Q}$.

Suppose (by contradiction) that $h + k\mathbf{t} \notin \mathcal{Q}$ for all k such that $0 < k < \omega$. We have $h \in M$. From Lemma 14 (i), if $h + \mathbf{t} \notin \mathcal{Q}$, then $h + \mathbf{t} \in \text{FLIP}(M + \mathbf{t}) = M$. Recursively, we have $h + k\mathbf{t} \in \text{FLIP}(M + \mathbf{t}) = M$ for all k with $0 < k < \omega$. This is summarized in the following graph:

$$\begin{array}{ccccccccccc} h & \xrightarrow{+\mathbf{t}} & h+\mathbf{t} & \xrightarrow{+\mathbf{t}} & h+2t & \xrightarrow{+\mathbf{t}} \cdots \xrightarrow{+\mathbf{t}} & h+(\omega-1)\mathbf{t} & \xrightarrow{+\mathbf{t}} & h+\omega\mathbf{t} \equiv h \\ \in \mathcal{Q} \cap M & & \in M \setminus \mathcal{Q} & & \in M \setminus \mathcal{Q} & & \in M \setminus \mathcal{Q} & & \in \mathcal{Q} \end{array}$$

But then, $h + (\omega - 1)\mathbf{t} \in M$ and $h + \omega\mathbf{t} \in \mathcal{Q}$, so that $h + \omega\mathbf{t} \notin \text{FLIP}(M + \mathbf{t}) = M$ from Lemma 14(i). This is a contradiction because $h + \omega\mathbf{t} \equiv h \in M$. Hence, there must exist some b_i, $0 < b_i < \omega$ such that $h + b_i\mathbf{t} \in \mathcal{Q}$. Since h is an edge parallel to the vector $\mathbf{e}_i$, then either $h + b_i\mathbf{t} \equiv (\mathbf{0}, \mathbf{e}_i)$ or $h + b_i\mathbf{t} \equiv (-\mathbf{e}_i, \mathbf{0})$. The first option contradicts the minimality of ω. We conclude that $\mathbf{e}_i + b_i\mathbf{t} \equiv \mathbf{0}$.

The number b_i is also unique. Indeed, suppose there exist b_i and b_i' with $0 < b_i < b_i' < \omega$ such that $h+b_i\mathbf{t} \equiv h+b_i'\mathbf{t} \equiv (-\mathbf{e}_i, \mathbf{0})$. Then $(b_i'-b_i)\mathbf{t} = (\mathbf{0}+b_i'\mathbf{t})-(\mathbf{0}+b_i\mathbf{t}) \equiv (-\mathbf{e}_i) - (-\mathbf{e}_i) = \mathbf{0}$. This contradicts the minimality of ω since $0 < b_i' - b_i < \omega$.

From the above paragraph, we have that $h + k\mathbf{t} \notin \mathcal{Q}$ for all k such that $0 < k < b_i$ or $b_i < k < \omega$. Using Lemma 14(i), if $h + (k - 1)\mathbf{t} \in M$ and $h + k\mathbf{t} \notin \mathcal{Q}$, then $h + k\mathbf{t} \in \text{FLIP}(M + \mathbf{t}) = M$. Thus, by recursion $h + k\mathbf{t} \in M$ for all k with $0 < k < b_i$. Also $h + b_i\mathbf{t} \equiv (-\mathbf{e}_i, \mathbf{0}) \in \mathcal{Q} \setminus M$. Using Lemma 14(ii), if $h + b_i\mathbf{t} \notin M$ and $h + (b_i + 1)\mathbf{t} \notin \mathcal{Q}$, then $h + (b_i + 1)\mathbf{t} \notin \text{FLIP}(M + \mathbf{t}) = M$. Thus by recursion $h + k\mathbf{t} \notin M$ for all k with $b_i < k < \omega$. □

Lemma 15 *$\mathbb{Z}^d/K$ is cyclic and generated by* $\mathbf{t}$.

Proof Let $\mathbf{u} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$. Using Proposition 5, we have

$$\mathbf{u} = \sum x_i\mathbf{e}_i \equiv \sum x_i(-b_i\mathbf{t}) = -\sum(b_i x_i)\mathbf{t}.$$

Let $k = -\sum(b_i x_i) \mod \omega$. Then, $0 \leq k < \omega$ and $\mathbf{u} \equiv k\mathbf{t}$. □

Lemma 16

$$M = \{(\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} : 1 \leq i \leq d \textit{ and } 0 \leq k < b_i\} + K.$$

Proof (⊇) If $0 \leq k < b_i$, then $(\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} \in M$ by Proposition 5.

($\subseteq$) Let $(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) \in M$ with $\mathbf{u} \in \mathbb{Z}^d$. From Lemma 15, $(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) = (\mathbf{0}, \mathbf{e}_i)+\mathbf{u} \equiv (\mathbf{0}, \mathbf{e}_i) + k\mathbf{t}$ for some integer k such that $0 \le k < \omega$. From Proposition 5, $0 \le k < b_i$. □

For all i with $1 \le i \le d$, let a_i be such that $a_i + b_i = \omega$. Also let

$$\mathbf{b} = (b_1, b_2, \dots, b_d) \quad \text{and} \quad \mathbf{a} = (a_1, a_2, \dots, a_d).$$

We have $a_i\mathbf{t} = (\omega - b_i)\mathbf{t} = \omega\mathbf{t} - b_i\mathbf{t} \equiv \mathbf{e}_i$. The next result shows that ω is a divisor of $\sum a_i$ and $\sum b_i$.

Lemma 17 *There exist integers q and ℓ, with $0 < q < d$ and $0 < \ell < d$ such that $\omega \cdot q = a_1 + a_2 + \cdots + a_d$ and $\omega \cdot \ell = b_1 + b_2 + \cdots + b_d$. Moreover, $d = q + \ell$.*

Proof For all $1 \le i \le d$, we have $\mathbf{e}_i = a_i\mathbf{t} = -b_i\mathbf{t}$. Thus, $-(b_1 + b_2 + \cdots + b_d)\mathbf{t}$ is an overall translation of $\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_d \in K$, i.e., the identity. Similarly, $(a_1 + a_2 + \cdots + a_d)\mathbf{t} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_d \in K$. Therefore, the order of $\mathbf{t}$ ($= \omega$) must divide both $a_1 + a_2 + \cdots + a_d$ and $b_1 + b_2 + \cdots + b_d$. Then, there exist integers q and ℓ such that $\omega \cdot q = a_1 + a_2 + \cdots + a_d$ and $\omega \cdot \ell = b_1 + b_2 + \cdots + b_d$. But $a_i < \omega$ for each i so that $a_1 + a_2 + \cdots + a_d < d\omega$ and $q < d$. Similarly, $\ell < d$. Finally, $\omega q + \omega\ell = \sum(a_i + b_i) = \omega d$ and, therefore, $d = q + \ell$. □

If the sum of the a_i or the sum of the b_i is ω, then the next two theorems claim that M is closely related to the Christoffel graph.

Theorem 2 (i) *If $\sum a_i = \omega$, then $M = \mathcal{H}_{\mathbf{a}}$;*
(ii) *if $\sum b_i = \omega$, then the complement $M^c = \mathbb{E}_d \setminus M$ of M is equal to $-\mathcal{H}_{\mathbf{b}}$.*

Proof (i) For all $\mathbf{u} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$, we have $\mathbf{u} \equiv k\mathbf{t}$ with

$$k = \sum(-b_i x_i) \bmod \omega = \sum(a_i - \omega)x_i \bmod \omega = \sum a_i x_i \bmod \sum a_i = \mathcal{F}_{\mathbf{a}}(\mathbf{u}).$$

We want to show that $M = \mathcal{H}_{\mathbf{a}}$. We have that $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) = (\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} \in M$ if and only if $0 \le k < b_i$ if and only if $\mathcal{F}_{\mathbf{a}}(\mathbf{u}) \in [0, \omega - a_i - 1]$ if and only if $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathcal{H}_{\mathbf{a}}$.

(ii) For all $\mathbf{u} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$, we have $\mathbf{u} \equiv k\mathbf{t}$ with

$$k = \sum(-b_i x_i) \bmod \omega = -\sum b_i x_i \bmod \sum b_i = -\mathcal{F}_{\mathbf{b}}(\mathbf{u}).$$

We want to show that $M^c = -\mathcal{H}_{\mathbf{b}}$. We have that $(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathbf{u}+\mathbf{e}_i) = (\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} \in \mathbb{E}_d \setminus M$ if and only if $b_i \le k < \omega$ if and only if $\mathcal{F}_{\mathbf{b}}(-\mathbf{u}) \in [b_i, \omega - 1]$ if and only if $(-\mathbf{u} - \mathbf{e}_i, -\mathbf{u}) \in \mathcal{H}_{\mathbf{b}}$ if and only if $(\mathbf{u}, \mathbf{v}) \in -\mathcal{H}_{\mathbf{b}}$. □

Corollary 4 *Let $d = 3$. M is the Christoffel graph $\mathcal{H}_{\mathbf{a}}$ or M is the complement of the reversal of the Christoffel graph $\mathcal{H}_{\mathbf{b}}$.*

Note that the complement of the reversal is equal to the reversal of the complement.

Proof From Lemma 17, there exist integers $0 < q < 3$ and $0 < \ell < 3$ such that $\omega \cdot q = a_1 + a_2 + a_3$ and $\omega \cdot \ell = b_1 + b_2 + b_3$. Therefore, there are two cases, either $q = 1$ and $\ell = 2$ or $q = 2$ and $\ell = 1$. If $q = 1$, then Theorem 2 (i) applies. Therefore, M is a Christoffel graph $M = \mathcal{H}_{\mathbf{a}}$ for the vector $\mathbf{a} = (a_1, a_2, a_3)$. If $\ell = 1$, then Theorem 2 (ii) applies. Therefore, the complement of M is the reversal of a Christoffel graph. More precisely, $M^c = -\mathcal{H}_{\mathbf{b}}$ for the vector $\mathbf{b} = (b_1, b_2, b_3)$. □

The previous result has also a counterpart in the triangular lattice L.

Corollary 5 *Let $M' \subset \pi(\mathbb{E}_d)$ such that* FLIP$(M' + \mathbf{t}') = M'$ *for some $\mathbf{t}' \in L$, that is invariant under some subgroup of finite index of L, and that satisfies $\mathcal{Q} \cap M = \{(\mathbf{0}, \mathbf{h}_1), (\mathbf{0}, \mathbf{h}_2), \ldots, (\mathbf{0}, \mathbf{h}_d)\} + K$. If $d = 3$, then M' is equal to a graph $I_{\mathbf{a}}$ or to the reversal of its edge-complement.*

Proof All we have to do is to lift M' to a set $M \subset \mathbb{E}_d$ using the projection $\pi : \mathbb{R}^d \to \mathcal{D}$ in such a way that $M =$ FLIP$(M + \mathbf{t})$, with $\pi(\mathbf{t}) = \mathbf{t}'$, and to show that M is invariant under the subgroup $K = \pi^{-1}(K')$ and satisfies $\mathcal{Q} \cap M = \{(\mathbf{0}, \mathbf{e}_1), (\mathbf{0}, \mathbf{e}_2), \ldots, (\mathbf{0}, \mathbf{e}_d)\} + K$. Then the corollary follows from the previous one. The details are left to the reader. □

Finally, we give the similar result for Christoffel parallelograms. We consider some parallelogram P whose vertices are in L, and edges (which are in $\mathbb{E}'_d$) within it; these edges must be compatible on a torus, in the sense that such an edge first hits the boundary of the parallelogram and then reappears on the opposite boundary of the parallelogram. Such a parallelogram defines a subgroup of finite index K' of L (spanned by the edges of the parallelogram) and tiles the whole hyperplane D. We say that FLIP$(P) = P + \mathbf{t}'$, for some $\mathbf{t}' \in L$, if $P + \mathbf{t}'$ is the parallelogram obtained by flipping the edges of P incident to zero mod K.

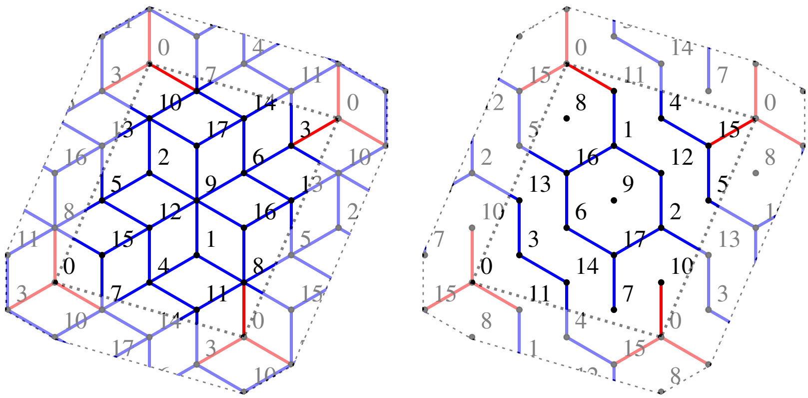
Corollary 6 *Under the previous hypothesis, P is a Christoffel parallelogram or the reversal of its edge-complement.*

Proof It suffices to verify that P defines a subset M' of $\pi(\mathbb{E}_d)$ satisfying the hypothesis of the previous corollary. □

The result is illustrated in Fig. 12.

We are now ready for the main result of this article, which generalizes Pirillo's theorem (Theorem 1) to an arbitrary dimension: a graph $M \subseteq \mathbb{E}_d$ is a translation of its flip if and only if it is a Christoffel graph of width ω.

Theorem 3 (d-dimensional Pirillo's theorem) *Let K be a subgroup of finite index of $\mathbb{Z}^d$ such that $\sum_{i=1}^{d} \mathbf{e}_i \in K$. Let $M \subseteq \mathbb{E}_d$ be a subset of edges invariant for the group of translations K such that the edges of M incident to zero mod K are $\mathcal{Q} \cap M = \{(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) \in \mathbb{E}_d \mid \mathbf{u} \in K\}$. There exists $\mathbf{t} \in \mathbb{Z}^d$ such that $M =$* FLIP$(M + \mathbf{t})$ *if and only if $M = \mathcal{H}_{\mathbf{a},\omega}$ is the Christoffel graph of normal vector $\mathbf{a}$ and width ω where ω is a divisor of $\|\mathbf{a}\|_1$ and $0 < \|\mathbf{a}\|_1/\omega < d$.*



$\mathcal{H}_{(3,7,8)}$ $\qquad$ $\mathcal{H}_{(15,11,10),18}$

Fig. 12 *Left* The Christoffel graph $\mathcal{H}_{\mathbf{a}}$ for the vector $\mathbf{a} = (3, 7, 8)$. It satisfies the equation $M = \text{FLIP}(M+\mathbf{t})$ for the translation vector $\mathbf{t} = \mathbf{e}_3 - \mathbf{e}_2$. *Right* The complement of the reversal of the Christoffel graph for the vector $\mathbf{b} = (3, 7, 8)$, i.e., $-\mathcal{H}_{\mathbf{b}}^c$. It corresponds to the Christoffel graph $\mathcal{H}_{\mathbf{a},\omega}$ for the vector $\mathbf{a} = (15, 11, 10)$ and width $\omega = 18$. It satisfies the equation $M = \text{FLIP}(M + \mathbf{t})$ for the translation vector $\mathbf{t} = \mathbf{e}_2 - \mathbf{e}_3$. They represent the only two possibilities for a pattern M satisfying $M = \text{FLIP}(M + \mathbf{t})$ when $d = 3$ and K is the subgroup of $\mathbb{Z}^3$ given by $\langle(0, 4, 1), (-2, 0, 3), (1, 1, 1)\rangle$

Proof Suppose $M = \text{FLIP}(M+\mathbf{t})$ for some $\mathbf{t} \in \mathbb{Z}^d$. Let ω be the order of $\mathbf{t}$ in the group $\mathbb{Z}^d/K$. From Lemma 15, for all $\mathbf{u} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}^d$, there exists an integer k such that $u \equiv k\mathbf{t}$ with

$$k = \sum(-b_i x_i) \bmod \omega = \sum(a_i - \omega)x_i \bmod \omega = \sum a_i x_i \bmod \omega = \mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}).$$

From Lemma 17, ω is a divisor of $\|\mathbf{a}\|_1$ and $0 < \|\mathbf{a}\|_1/\omega < d$. We want to show that $M = \mathcal{H}_{\mathbf{a},\omega}$. We have that $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i) = (\mathbf{0}, \mathbf{e}_i) + k\mathbf{t} \in M$ if and only if $0 \le k < b_i$ if and only if $\mathcal{F}_{\mathbf{a},\omega}(\mathbf{u}) \in [0, \omega - a_i - 1]$ if and only if $(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) \in \mathcal{H}_{\mathbf{a},\omega}$ from Lemma 9.

Reciprocally, suppose $\mathcal{H}_{\mathbf{a},\omega}$ is the Christoffel graph of normal vector $\mathbf{a}$ of width ω. We can show that $\mathcal{H}_{\mathbf{a},\omega} + \mathbf{t} = \text{FLIP}(\mathcal{H}_{\mathbf{a},\omega})$ where $\mathbf{t} \in \mathbb{Z}^d$ is such that $\mathcal{F}_{\mathbf{a},\omega}(\mathbf{t}) = 1$. The proof goes along the same lines as Proposition 4 using Lemma 9 instead of Lemma 3. □

Remark 2 The third hypothesis introduced at the beginning of the section saying that the legs of M are positive, i.e., $\mathcal{Q} \cap M = \{(\mathbf{u}, \mathbf{u}+\mathbf{e}_i) \in \mathbb{E}_d \mid \mathbf{u} \in K\}$, can be removed. The graph $M \subseteq \mathbb{E}_d$ that we obtain satisfying the first two hypotheses is related to Christoffel graphs. If $\sum_{i=1}^{d} \mathbf{e}_i \in K$, then we may use the same function $\mathcal{F}_{\mathbf{a}}$ defined on a positive normal vector $\mathbf{a} \in \mathbb{N}^d$. One can show that the graph M is the union of the set of edges $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i)$ that are increasing for the function $\mathcal{F}_{\mathbf{a}}$ when $(\mathbf{0}, e_i) \in M$ and the set of edges $(\mathbf{u}, \mathbf{u} + \mathbf{e}_i)$ that are decreasing for the function $\mathcal{F}_{\mathbf{a}}$ when $(-\mathbf{e}_i, \mathbf{0}) \in M$.

Acknowledgments We wish to thank the anonymous referee for his many valuable comments and for having noticed that Lemma 13 was missing. Both authors acknowledge the support of the Natural Sciences and Engineering Research Council of Canada (NSERC). The first author acknowledges support from the ANR project Dyna3S (ANR-13-BS02-0003). Sage open source software was used to generate `tikz` code to create the artwork.

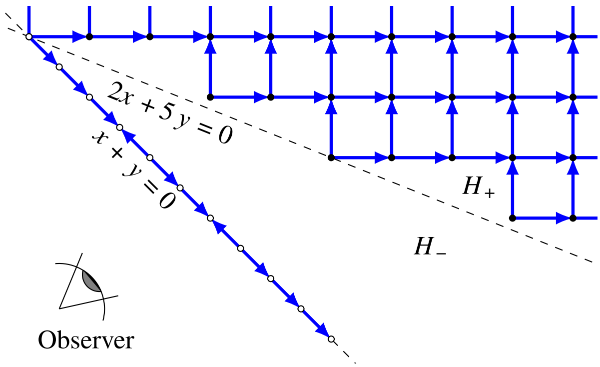


Fig. 13 Observation in dimension 2. What the observer sees can be projected parallel to the vector (1, 1) on the line $x + y = 0$

Appendix: Digital Planes

In this section, we show some results on standard digital planes. Digital planes were introduced in [25] and standard digital planes were further studied in [19]. The projection of a standard digital plane gives a tiling of $\mathcal{D}$ by three kinds of rhombus [8], thus yielding a coding of it by $\mathbb{Z}^2$-actions by rotations on the unit circle [3,4]. Our construction of the discretized hyperplane is equivalent, for the dimension 3, to that in [3]. Our point of view is slightly different from the classical one; inspired by the 2-dimensional case (digital lines), we define a digital hyperplane by "what the observer sees": the observer is at $-\infty$ in the direction $(1, 1, \ldots, 1)$ and he looks towards the "transparent" hyperplane of all the unit hypercubes, which are located on the other side. This may be modeled mathematically; all the results are intuitively clear, but require a proof. We prove them, since we could not find precise references. We recover some known results.

Imagine the d-dimensional space filled with unit hypercubes with opaque faces. Consider a hyperplane H of equation $\sum a_i x_i = 0$, $a_i > 0$ coprime integers. An observer sits in the open half-space H_- bounded by the hyperplane. Then, we remove all the cubes in this half-space, including the cubes intersecting this half-space; in other words, we keep only the cubes contained in H_+. Figure 13 illustrates this construction for $d = 2$.

For $d = 3$, when we look towards H_+ parallel to the vector $(1, 1, 1)$, we see something like in Fig. 14.

Let s be the sum $s = \sum_i a_i$. We denote $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d$. The complement of H has two connected components H_- and H_+, where the first is determined by the inequality $\sum_i a_i x_i < 0$. We consider the unit cubes in $\mathbb{R}^d$ and their facets. Such a facet is a subset of $\mathbb{R}^d$ of the form $M + \sum_{j \neq i} [0, 1]\mathbf{e}_j$ for some coordinate $i \in \{1, \ldots, d\}$ and some integral point $M \in \mathbb{Z}^d$. Denote by $\mathcal{C}$ the standard unit cube.

Consider the unit hypercubes that are contained in the closed half-space $H \cup H_+$ and their facets; denote by $\mathcal{U}_+$ the union of all these facets. Note that a unit cube

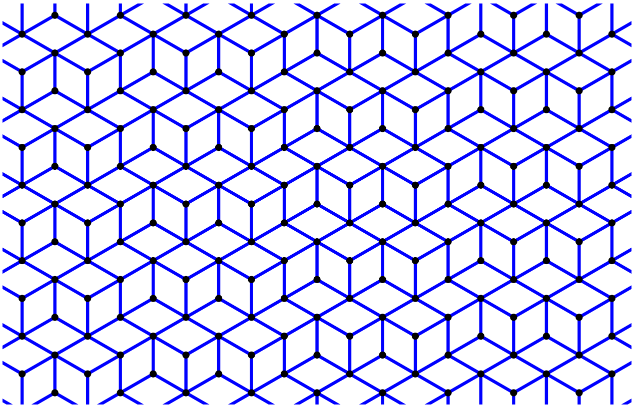


Fig. 14 What the observer sees in dimension 3. The surface of cubes was projected parallel to the vector (1, 1, 1) on the plane $x + y + z = 0$

$M + \mathcal{C}$ ($M \in \mathbb{Z}^d$) is contained in $H \cup H_+$ if and only if $M \in H \cup H_+$ if and only if $\sum_j a_j m_j \geq 0$.

We say that a point M is in $\mathbb{R}^d$ is *visible* if the open half-line $\{M - (x, x, \ldots, x) \mid x > 0\}$ does not contain any point in $\mathcal{U}_+$. Intuitively, this means, all facets being opaque, that an observer located at infinity in the direction of the vector $-(1, 1, \ldots, 1)$ can see this point M, because no point in $\mathcal{U}_+$ hides this point.

Now, we consider the set of visible points, which belong to $\mathcal{U}_+$. This we may call the *discretized hyperplane associated with* H. Intuitively, it is the set of facets that the observer can see, as is explained in the introduction.

We characterize now the discretized hyperplane. For this, we denote by R the following subset of $\mathbb{R}^d$: $R = \{(x_i) \mid 0 \leq \sum_i a_i x_i < s\}$. Note that $R \subset H \cup H_+$.

Denote by $\mathcal{S}$ the union of the facets that are contained in R. In other words,

$$\mathcal{S} = \bigcup_{M \in \mathbb{Z}^d, 1 \leq i \leq d, M + \sum_{j \neq i}[0,1]\mathbf{e}_j \subset R} \Big(M + \sum_{j \neq i} [0, 1]\mathbf{e}_j\Big).$$

Observe that the condition $M + \sum_{j \neq i}[0, 1]\mathbf{e}_j \subset R$ is equivalent to $\sum_j a_j m_j \geq 0$ and $a_i m_i + \sum_{j \neq i} a_j (m_j + 1) < s$. Note also that $\mathcal{S} \subset \mathcal{U}_+$.

Theorem 4 *The discretized hyperplane is equal to* $\mathcal{S}$.

Observe that if we project $\mathcal{S}$ onto the hyperplane perpendicular to the vector $(1, 1, \ldots, 1)$, we obtain exactly what the observer sees.

An example of this, for $d = 3$, is given in Fig. 14. This observation motivates the introduction of the graph $I_{\mathbf{a}}$ in Sect. 3.2.

We first give a simple characterization of $\mathcal{S}$.

Proposition 6 *Let* $X = (x_i) \in \mathbb{R}^d$. *Then* X *is in* $\mathcal{S}$ *if and only if the following three conditions hold:*

(i) *some coordinate of* X *is an integer;*

(ii) $\sum_i a_i \lfloor x_i \rfloor \geq 0$;
(iii) $\sum_i a_i \lceil x_i \rceil < s$.

We recover Proposition 1 of [3].

Corollary 7 *Let $X = (x_i) \in \mathbb{Z}^d$. Then X is in $\mathcal{S}$ if and only if $0 \leq \sum_i a_i x_i < s$.*

Proof of the Proposition Suppose that $X \in \mathcal{S}$. Then $X \in M + \sum_{j \neq i} [0, 1]\mathbf{e}_j \subset \mathcal{S}$ and the coordinates m_j of M are integers. Thus, by an observation made previously, $0 \leq \sum_j a_j m_j \leq \sum_j a_j \lfloor x_j \rfloor$, since $x_j = m_j + \theta_j$, with $0 \leq \theta_j \leq 1$ and $\theta_i = 0$. Moreover, $\lceil x_i \rceil = m_i$, and $\lceil x_j \rceil \leq m_j + 1$ if $j \neq i$. Thus, $\sum_j a_j \lceil x_j \rceil \leq a_i m_i + \sum_{j \neq i} (m_j + 1) < s$, by the same observation.

Conversely, suppose that the three conditions of the proposition hold. Without restricting the generality (by permutation of the coordinates), we may assume that for some $i \in \{1, \ldots, d\}$, one has $x_1, \ldots, x_i \in \mathbb{Z}$ and $x_{i+1}, \ldots, x_d \notin \mathbb{Z}$. Let $0 \leq p \leq i$ be maximum subject to the condition $\sum_{j \leq p} a_j (x_j - 1) + \sum_{j > p} a_j \lfloor x_j \rfloor \geq 0$ (note that p exists since the inequality is satisfied for $p = 0$). Suppose by contradiction that $p = i$; then $\sum_j a_j \lceil x_j \rceil = \sum_{j \leq i} a_j x_j + \sum_{j > i} a_j (\lfloor x_j \rfloor + 1) = a_1 + \cdots + a_d + \sum_{j \leq i} a_j (x_j - 1) + \sum_{j > i} a_j \lfloor x_j \rfloor \geq a_1 + \cdots + a_d$ (since $p = i$) $= s$; thus, we obtain a contradiction with condition (iii).

Thus, $p < i$ and $p+1 \leq i$. We have by maximality the inequality $\sum_{j \leq p+1} a_j (x_j - 1) + \sum_{j > p+1} a_j \lfloor x_j \rfloor < 0$. Let $M = (m_j) = (x_1 - 1, \ldots, x_p - 1, \lfloor x_{p+1} \rfloor, \ldots, \lfloor x_d \rfloor \in \mathbb{Z}^d$. We have $\sum_j a_j m_j \geq 0$ (by definition of p) and $a_{p+1} m_{p+1} + \sum_{j \neq p+1} a_j (m_j + 1) = \sum_{j \leq p} (a_j (x_j - 1) + a_j) + a_{p+1} (x_{p+1} - 1) + a_{p+1} + \sum_{j > p+1} (a_j \lfloor x_j \rfloor + a_j) = s + \sum_{j \leq p+1} a_j (x_j - 1) + \sum_{j > p+1} a_j \lfloor x_j \rfloor < s$, by the previous inequality. Thus, $M + \sum_{j \neq i} [0, 1]\mathbf{e}_j \subset \mathcal{S}$, by the observation made above. Moreover, $X \in M + \sum_{j \neq i} [0, 1]\mathbf{e}_j$ since $p + 1 \leq i$. □

Corollary 8 *For each point X in $\mathbb{R}^d$, there is a unique point Y in $\mathcal{S}$ such that XY is parallel to the vector* $(1, 1, \ldots, 1)$.

Denote by f the function such that $Y = f(X)$ with the notations of the corollary. This function is a kind of projection onto $\mathcal{S}$, parallel to the vector $(1, 1, \ldots, 1)$. Denote also by $t(X)$ the real-valued function defined by $X = f(X) + t(X)(1, 1, \ldots, 1)$.

Proof We prove first unicity. By contradiction, we have $Y, Z \in \mathcal{S}$ and $Z = Y + t(1, 1, \ldots, 1)$ with $t > 0$. Then $z_i = y_i + t$. Thus $\lceil z_i \rceil \geq \lfloor y_i \rfloor + 1$. Hence, $\sum_i a_i \lceil z_i \rceil \geq s + \sum_i a_i \lfloor y_i \rfloor$. Since by Proposition 6, applied to Y, the last sum is ≥ 0, we obtain $\sum_i a_i \lceil z_i \rceil \geq s$, which contradicts Proposition 6, applied to Z.

We now prove the existence of Y. Define $L(X) = \sum_i a_i \lfloor x_i \rfloor$; we may assume that $L(X) \geq 0$, by adding to X some positive multiple of $(1, 1, \ldots, 1)$ if necessary. We prove the existence of Y by induction on the sum $U(X) = \sum_i a_i \lceil x_i \rceil$.

Let $\varepsilon = \min_i (x_i - \lfloor x_i \rfloor)$. Observe that if we replace X by $X - \varepsilon(1, 1, \ldots, 1)$, then $L(X)$ does not change, $U(X)$ does not increase and; moreover, some x_i is now an integer.

If $U(X)$ is smaller than s, this observation implies the existence of Y.

Suppose now that $U(X) \geq s$. By the observation, we may assume that at least one of the x_i is an integer. Without restricting the generality, we may also assume that $x_1, \ldots, x_i \in \mathbb{Z}$ and that $x_{i+1}, \ldots, x_d \notin \mathbb{Z}$, with $i \geq 1$.

If $i = d$, then the x_j are all integers, $L(X) = U(X)$; we replace X by $X - (1, 1, \ldots, 1)$ and we conclude by induction, since $L(X)$ is replaced by $L(X) - s$.

Suppose now that $i < d$. Let $\varepsilon = \min_{j>i}(x_j - \lfloor x_j \rfloor)$; then $\varepsilon > 0$. We have $s \le \sum_j a_j \lceil x_j \rceil = \sum_{j \le i} a_j x_j + \sum_{j>i} a_j(\lfloor x_j \rfloor + 1) = L(X) + a_{i+1} + \cdots + a_d$, hence $L(X) \ge a_1 + \cdots + a_i$. Note that $\sum_j a_j(\lfloor x_j - \varepsilon \rfloor) = \sum_{j \le i} a_j(x_j - 1) + \sum_{j>i} a_j \lfloor x_j \rfloor = L(X) - a_1 - \cdots - a_i \ge 0$. We replace X by $X - \varepsilon(1, 1, \ldots, 1)$, and we may conclude by induction, since $U(X)$ strictly decreases and since $L(X)$ remains ≥ 0. □

Proof of the Theorem Let X be a point on the discretized hyperplane associated with H. Suppose that $t(X) > 0$. Then $X = f(X) + t(X)(1, 1, \ldots, 1)$ so that X is hidden by $f(X)$: formally, $f(X)$ is on the open half-line $\{X - (x, x, \ldots, x) \mid x > 0\}$, and since $f(X)$ is in $\mathcal{S}$, it is a point in $\mathcal{U}_+$. We conclude that we must have $t(X) \le 0$. Suppose that $t(X) < 0$. We know that X is in $\mathcal{U}_+$ so that X belongs to a hypercube $M + \mathcal{C}$ with $\sum_j a_j m_j \ge 0$, and, therefore, $x_j \ge m_j$. Let $Y = f(X)$. Then $X = Y + t(X)(1, 1, \ldots, 1)$ so that $y_j > x_j \ge m_j$, which implies $\sum_j a_j \lceil y_j \rceil \ge \sum_j a_j(m_j + 1) \ge s$, a contradiction with Proposition 6. Thus, $t(X) = 0$ and $X \in \mathcal{S}$.

Conversely suppose that $X \in \mathcal{S}$. Suppose that X is not on the discretized hyperplane associated to H. This implies that there is some point $Y \in \mathcal{U}_+$ on the open half-line $\{X - (x, x, \ldots, x) \mid x > 0\}$. We have $Y \in M + \mathcal{C}$ with $\sum_j a_j m_j \ge 0$. Thus, $x_j > y_j \ge m_j$,which implies that $\sum_j a_j \lceil x_j \rceil \ge \sum_j a_j(m_j + 1) \ge s$, a contradiction with Proposition 6. □

Corollary 9 *Let $d \ge 2$. Let $M \in \mathcal{S} \cap \mathbb{Z}^d$. Let $i = 1, 2, \ldots, d$ and $N = M + \mathbf{e}_i$.*

(i) *$N \in \mathcal{S}$ if and only if $\sum_j a_j n_j < s$; in this case, the segment $M + [0, 1]\mathbf{e}_i$ is contained in $\mathcal{S}$.*
(ii) *If $N \notin \mathcal{S}$, then the only point in $(M + [0, 1]\mathbf{e}_i) \cap \mathcal{S}$ is M.*

Proof The fact that $N \in \mathcal{S}$ if and only if $\sum_j a_j n_j < s$ is a consequence of the proposition.

Suppose that $N \in \mathcal{S}$ and let X be on the segment $M + [0, 1]\mathbf{e}_i$. Then $0 \le \sum_j a_j m_j \le \sum_j a_j \lfloor x_j \rfloor$ and $\sum_j a_j \lceil x_j \rceil \le \sum_j a_j n_j < s$. Thus, the corollary follows from the proposition.

Suppose that $N \notin \mathcal{S}$ and let X be on this segment. Since $0 \le \sum_j a_j m_j$, we have also $0 \le \sum_j a_j n_j$. Since $N \notin \mathcal{S}$, we must have $\sum_j a_j n_j \ge s$. Moreover, if $X \ne M$, we have $\lceil x_j \rceil = n_j$ so that $\sum_j a_j \lceil x_j \rceil \ge s$ and $X \notin \mathcal{S}$. □

The next result, which is not needed in this article, is of independent interest, and intuitively clear (but it requires a proof).

Proposition 7 *The function $f : X \mapsto Y$, with the notations of Corollary 8, is continuous. The open set $\mathbb{R}^d \setminus \mathcal{S}$ has two connected components.*

Lemma 18 *Let $\mathcal{S}$ be a closed subset of $\mathbb{R}^d$such that for each X in $\mathbb{R}^d$, there is a unique Y in $\mathcal{S}$ such that XY is parallel to $(1, 1, \ldots, 1)$. If the mapping $f : X \mapsto Y$ is bounded (that is, the image of each bounded set is bounded), then it is continuous.*

Proof Recall that a bounded sequence in $\mathbb{R}^d$ converges if for any two convergent subsequences, they have the same limit. Let (X_n) be a sequence in $\mathbb{R}^d$, with limit

l. It is enough to show that $(f(X_n))$ converges; note that this sequence is bounded. Consider two subsequences of (X_n) such that their images under f have limits, l_1 and l_2, say. Since $\mathcal{S}$ is closed, $l_1, l_2 \in \mathcal{S}$. Let $\varepsilon > 0$. For n large enough, $|X_n - l| < \varepsilon$; hence, $f(X_n)$ is in the open cylinder of diameter ε with the line $l + (1, 1, ..., 1)$ as its central axis. This implies that l_1, l_2 are in this cylinder and, consequently, ε being arbitrary, l_1, l_2 are on the previous line. By unicity, $l_1 = l_2$ $(=f(l))$. We conclude using the remark at the beginning of the proof. □

Proof of the Proposition The mapping f is continuous: by Lemma 18, it is enough to show that $\mathcal{S}$ is closed and that the mapping is bounded. Since each convergent sequence is contained in some compact set, it is enough to show that for each compact set K, $K \cap \mathcal{S}$ is closed; but this is clear, since the latter set is the union of finitely many $K \cap F$, F facet of a unit hypercube. The mapping is bounded since its image is between the two hyperplanes of equations $\sum_i a_i x_i = 0$ and $\sum_i a_i x_i = s$, so that the image of each bounded set is contained in a cylinder of axis parallel to $(1, 1, \ldots)$ and limited by these two hyperplanes.

Now, we show that the set $\mathbb{R}^d \setminus \mathcal{S}$ has two connected components. Note that for each point X, one has $X = f(X) + t(X)(1, 1, \ldots, 1)$ for some continuous real-valued function t. Since $f(1, 1, \ldots, 1) = (0, 0, \ldots, 0) = f(-1, -1, \ldots, -1)$, one has $t(1, 1, \ldots, 1) = 1$ and $t(-1, -1, \ldots, -1) = -1$. Moreover, $t(X) = 0$ if and only if $X \in \mathcal{S}$. Thus, $t(\mathbb{R}^d \setminus \mathcal{S})$ is not connected and neither is $\mathbb{R}^d \setminus \mathcal{S}$.

Now, if $t(X) > 0$, one may connect X by a piece of the line $X + \mathbb{R}(1, 1, \ldots, 1)$ to a point of the half-space $\sum_i a_i x_i > 0$ and this implies that the set of points X with $t(X) > 0$ is connected. Similarly, the set of points with $t(X) < 0$ is connected, and $\mathbb{R}^d) \setminus \mathcal{S}$ has, therefore, two connected components. □

We recover Proposition 2 of [3] and Proposition 4 of [4].

Corollary 10 *The restriction of f to $\mathcal{D}$ is a homeomorphism of $\mathcal{D}$ onto $\mathcal{S}$.*

Proof Indeed, the inverse mapping is the projection onto the hyperplane $\mathcal{D}$ parallel to the vector $(1, 1, \ldots, 1)$. □

References

1. Andres, E., Raj, A., Claudio, S.: Discrete analytical hyperplanes. Graph. Models Image Process. **59**(5), 302–309 (1997)
2. Arnoux, P., Berthé, V., Ei, H., Ito, S.: Tilings, quasicrystals, discrete planes, generalized substitutions, and multidimensional continued fractions. Discrete Models: Combinatorics, Computation, and Geometry (Paris, 2001). Discrete Mathematics & Theoretical Computer Science Proceedings, AA, pp. 059–078. Maison Inform. Math. Discrèt. (MIMD), Paris (2001)
3. Arnoux, P., Berthé, V., Ito, S.: Discrete planes, $\mathbb{Z}^2$-actions, Jacobi–Perron algorithm and substitutions. Ann. Inst. Fourier (Grenoble) **52**(2), 305–349 (2002)
4. Arnoux, P., Berthé, V., Fernique, T., Jamet, D.: Functional stepped surfaces, flips, and generalized substitutions. Theor. Comput. Sci. **380**(3), 251–265 (2007). doi:10.1016/j.tcs.2007.03.031
5. Berstel, J.: Sturmian and episturmian words. In: Bozapalidis, S., Rahonis, G. (eds.) Algebraic Informatics. Lecture Notes in Computer Science, vol. 4728, pp. 23–47. Springer, Berlin (2007). doi:10.1007/978-3-540-75414-5_2
6. Berstel, J., Lauve, A., Reutenauer, C., Saliola, F.: Combinatorics on Words: Christoffel Words and Repetition in Words. CRM Monograph Series, vol. 27. American Mathematical Society, Providence, RI (2008)

7. Berthé, V., Tijdeman, R.: Lattices and multi-dimensional words. Theor. Comput. Sci. **319**(1–3), 177–202 (2004). doi:10.1016/j.tcs.2004.02.016
8. Berthé, V., Vuillon, L.: Tilings and rotations on the torus: a two-dimensional generalization of Sturmian sequences. Discrete Math. **223**(1–3), 27–53 (2000). doi:10.1016/S0012-365X(00)00039-X
9. Bodini, O., Fernique, T., Rémila, É.: A characterization of flip-accessibility for rhombus tilings of the whole plane. Inf. Comput. **206**(9–10), 1065–1073 (2008). doi:10.1016/j.ic.2008.03.008
10. Bodini, O., Fernique, T., Rao, M., Rémila, É.: Distances on rhombus tilings. Theor. Comput. Sci. **412**(36), 4787–4794 (2011). doi:10.1016/j.tcs.2011.04.015
11. Borel, J.P., Reutenauer, C.: On Christoffel classes. Theor. Inform. Appl. **40**(1), 15–27 (2006). doi:10.1051/ita:2005038
12. Brimkov, V., Coeurjolly, D., Klette, R.: Digital planarity—a review. Discrete Appl. Math. **155**(4), 468–495 (2007). doi:10.1016/j.dam.2006.08.004
13. Brlek, S., Hamel, S., Nivat, M., Reutenauer, C.: On the palindromic complexity of infinite words. Int. J. Found. Comput. Sci. **15**(2), 293–306 (2004)
14. Carpi, A., Luca, A.: Central Sturmian words: recent developments. In: Felice, C., Restivo, A. (eds.) Developments in Language Theory. Lecture Notes in Computer Science, vol. 3572, pp. 36–56. Springer, Berlin (2005). doi:10.1007/11505877_4
15. Chuan, W.F.: α-Words and factors of characteristic sequences. Discrete Math. **177**(1–3), 33–50 (1997). doi:10.1016/S0012-365X(96)00355-X
16. Debled-Rennesson, I.: Reconnaissance des droites et plans discrets. Université Louis Pasteur - Strasbourg, Thèse de Doctorat (1995)
17. Domenjoud, E., Vuillon, L.: Geometric palindromic closure. Unif. Distrib. Theory **7**(2), 109–140 (2012)
18. Fernique, T.: Pavages, fractions continues et géométrie discrète. Thèse de Doctorat, Université Montpellier 2 (2007). http://tel.archives-ouvertes.fr/tel-00206966
19. Françon, J.: Sur la topologie d'un plan arithmétique. Theor. Comput. Sci. **156**(1–2), 159–176 (1996). doi:10.1016/0304-3975(95)00059-3
20. Françon, J., Schramm, M., Tajine, M.: Recognizing arithmetic straight lines and planes. Discrete Geometry for Computer Imagery (Lyon, 1996). Lecture Notes in Computer Science, vol. 1176, pp. 141–150. Springer, Berlin (1996)
21. Ito, S., Ohtsuki, M.: Modified Jacobi–Perron algorithm and generating Markov partitions for special hyperbolic toral automorphisms. Tokyo J. Math. **16**(2), 441–472 (1993). doi:10.3836/tjm/1270128497
22. Ito, S., Ohtsuki, M.: Parallelogram tilings and Jacobi–Perron algorithm. Tokyo J. Math. **17**(1), 33–58 (1994). doi:10.3836/tjm/1270128186
23. Pirillo, G.: A curious characteristic property of standard Sturmian words. Algebraic Combinatorics and Computer Science, pp. 541–546. Springer, Milan (2001)
24. Provot, L., Buzer, L., Debled-Rennesson, I.: Recognition of blurred pieces of discrete planes. In: Discrete Geometry for Computer Imagery. Lecture Notes in Computer Science, vol. 4245, pp. 65–76. Springer, Berlin (2006). doi:10.1007/11907350_6
25. Reveillès, J.P.: Géométrie discrète, calcul en nombres entiers et algorithmique. Thèse de Doctorat, Université Louis Pasteur, Strasbourg (1991)
26. Reveillès, J.P.: Combinatorial pieces in digital lines and planes. In: Vision Geometry, IV (San Diego, CA, 1995). Proceedings of SPIE, vol. 2573, pp. 23–34. SPIE, Bellingham (1995). doi:10.1117/12.216425
27. Vuillon, L.: Local configurations in a discrete plane. Bull. Belg. Math. Soc. Simon Stevin **6**(4), 625–636 (1999)