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A bijection between words and multisets of necklaces[☆]

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ABSTRACT

Two of the present authors have given in 1993 a bijection Φ between words on a totally ordered alphabet and multisets of primitive necklaces. At the same time and independently, Burrows and Wheeler gave a data compression algorithm which turns out to be a particular case of the inverse of Φ. In the present article, we show that if one replaces in Φ the standard permutation of a word by the co-standard one (reading the word from right to left), then the inverse bijection is computed using the alternate lexicographic order (which is the order of real numbers given by continued fractions) on necklaces, instead of the lexicographic order as for Φ^{-1}. The image of the new bijection, instead of being as for Φ the set of all multisets of primitive necklaces, is a special set of multisets of necklaces (not all primitive); it turns out that this set is naturally linked to the decomposition of the enveloping algebra of the oddly generated free Lie superalgebra, induced by the Poincaré–Birkhoff–Witt theorem.

1. Introduction

In [4], it has been proved the bijectivity of a natural mapping Φ from words on a totally ordered alphabet onto multisets of primitive necklaces (circular words) on this alphabet. This mapping has many enumerative applications; among them, the fact that the number of permutations in a given conjugation class and with a given descent set is equal to the scalar product of two representations naturally associated to the class and the set.

☆ To our friend Toni Machí.

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The direct mapping is defined by associating to each word its *standard permutation*, and then replacing in the cycles of the latter each element by the corresponding letter of the word. The inverse mapping Φ^{-1} is constructed using the lexicographic order of infinite words.

In the present article, we replace the standard permutation by the *costandard permutation*. That is, the permutation obtained by numbering the positions in the word from right to left (instead from left to right as it is done for the standard permutation). This a priori useless generalization has however striking properties. Indeed, it induces a bijection Ξ between words and multisets of necklaces, which are intimately related to the Poincaré–Birkhoff–Witt theorem applied to the free Lie superalgebra (instead of the free Lie algebra as it is the case for Φ). See Section 3 for the exact description of these multisets.

Another striking property of the bijection Ξ is that the inverse bijection uses, instead of the lexicographic order, the *alternate lexicographic order* of infinite words; this means that one compares the first letters of the infinite words for the given order of the alphabet, then if equality, the second letters for the opposite order, and so on.

The alternate lexicographic order is very natural. Indeed it corresponds to the order of real numbers given by their expansion into continued fractions. This is well-known and used implicitly very often; see for example the book [3] on the Markoff and Lagrange spectra.

The proof of the bijectivity of Ξ that we give has as a byproduct a new proof of the bijectivity of Φ, that we give in Section 2. In Section 5 we recall the symmetric functions that are induced by Φ, and then show that the symmetric functions induced by Ξ are related to the free Lie superalgebra and are equal to the image of the formers under the fundamental involution of symmetric functions.

2. Reminder: the lexicographic bijection

Recall that the *standard permutation* of a word $w = a_1 \cdots a_n$ of length n on a totally ordered alphabet A is the permutation σ, denoted $st(w)$, such that for any $i, j \in \{1, \ldots, n\}$, the condition $\sigma(i) < \sigma(j)$ is equivalent to $a_i < a_j$ or $a_i = a_j$ and $i < j$. The permutation $st(w)$ may be obtained by numbering from left to right the letters of w, starting with the smallest letter, then the second smallest, and so on. Equivalently, $st(w)$ is the unique shortest permutation $\sigma \in S_n$ such that $w.\sigma^{-1}$ is an increasing word. Here, the length $l(\sigma)$ of a permutation σ is as usually the number of its inversions and $w.\tau$ denotes the right action of the permutation $\tau \in S_n$ on the word w, defined by: $w.\tau = a_{\tau(1)} \cdots a_{\tau(n)}$. As an example, we have $st(baacbcab) = 41275836$, $41275836^{-1} = 23715846$ and $baacbcab.23715846 = aaabbbcc$.

Two words are *conjugate* if they may be written uv and vu for some words u, v. A conjugation class is called a *necklace*, or a *circular word*. It is *primitive* if the words in the class are not a nontrivial power of another word; equivalently, the circular word, viewed as regular n-gon with labels in A, is not fixed by any nontrivial rotation. If w is a word, we denote the corresponding circular word by (w), imitating the word and cycle notation of permutations.

We describe now the bijection Φ of [4] between the set A^* of words on A and the set **M** of multisets of primitive necklaces on A. Actually, we present a slight variant of it (replacing standard permutation by its inverse) in order to make it compatible with the Burrows–Wheeler transform [1] (see also [2,8]). This is not an essential change.

Let $w = a_1 \cdots a_n$ be a word and τ the inverse of its standard permutation. With each cycle $(j_1, \ldots, j_i)$ of τ, associate the necklace $(a_{j_1}, \ldots, a_{j_i})$. It turns out that this necklace is primitive. The mapping Φ which associates with w the multiset of necklaces obtained by taking all cycles of τ is bijective. Consider the example above. The cycles of τ are $(1, 2, 3, 7, 4)$, (5) and $(6, 8)$. The corresponding multiset of necklaces is therefore $\Phi(w) = \{(baaac), (b), (cb)\}$.

In order to describe the inverse bijection, we follow [7]. Given a multiset of primitive necklaces, consider the corresponding multiset of words (there are i words for each necklace of length i); note that these words are all primitive. Put them in order, from top to bottom, in a right justified tableau, where the order is as follows: $u < v$ if and only if $uuuu \cdots < vvvv \cdots$ (lexicographic order for infinite words). Note that if there are nontrivial multiplicities in the multiset, then there are repeated rows. Then the inverse image of the multiset is the last column, read from top to bottom. As an example, take the previous multiset $\{(baaac), (b), (cb)\}$. The multiset of words is $\{baaac, aaacb, aacba, acbaa, cbaaa, b, cb, bc\}$. The order on the corresponding infinite words is $aaacb\cdots < aacba\cdots < acbaa\cdots < baaac\cdots < bb\cdots < bc\cdots < cbaaa\cdots < cbcb\cdots$. Thus the tableau is

a	a	a	c	b
a	a	c	b	a
a	c	b	a	a
b	a	a	a	c
				b
			b	c
c	b	a	a	a
			c	b

Its last column is *baacbcab* as desired.

We give now a new proof of the bijectivity of Φ. It is simpler than the proof in [4]. Let w, τ be as above.

1. If $p < q$, then $a_{\tau(p)} \leq a_{\tau(q)}$: indeed, this follows from the definition of $st(w)$ by numbering from left to right and from the fact that τ is the inverse of $st(w)$.
2. If $p < q$ and if $a_{\tau(p)} = a_{\tau(q)}$, then $\tau(p) < \tau(q)$: the definition of $st(w)$ by numbering from left to right shows that if $i < j$ and $a_i = a_j$, then $st(i) < st(j)$. Suppose now that $p < q$ and $a_{\tau(p)} = a_{\tau(q)}$. Put $i = \tau(p)$ and $j = \tau(q)$; then $a_i = a_j$; if we had $\tau(p) > \tau(q)$, that is, $i > j$, we would have $st(i) > st(j)$, that is, $p > q$, a contradiction. Thus $\tau(p) < \tau(q)$.
3. For $p = 1, \ldots, n$, let $w_p = a_{\tau(p)} a_{\tau^2(p)} \ldots a_{\tau^k(p)}$ where k is the length of the cycle of τ containing p. Note that $\tau^k(p) = p$ and therefore the last letter of w_p is a_p.
4. The p-th row of the tableau associated to the multiset $\Phi(w)$ is w_p: this is a consequence of the fact that if $p < q$, then $w_p^\infty < w_q^\infty$ (lexicographical order), which follows from 1 and 2, observing that $w_p^\infty = a_{\tau(p)} a_{\tau^2(p)} a_{\tau^3(p)} \cdots$.
5. It follows that the last column of this tableau is w, which proves the injectivity of Φ.
6. If $p < q$ and $w_p = w_q$, then $\tau^i(p) < \tau^i(q)$: indeed, since $w_p = w_q$, we have $w_{\tau^i(p)} = w_{\tau^i(q)}$. Since moreover $p < q$, by 2, $\tau(p) < \tau(q)$ (since $w_p = w_q$ implies that their first letters coincide, that is, $a_{\tau(p)} = a_{\tau(q)}$). This proves that $\tau^i(p) < \tau^i(q)$ by induction on i.
7. Suppose that some w_p is not primitive. Then $w_p = u^l$, $l \geq 2$. Let $q = \tau^h(p)$, where h is the length of u. Then $w_p = w_q$. Moreover $p \neq q$ since $h < |w_p|$ = length of the cycle of τ at p. Suppose that $p < q$ (the case $p > q$ is similar). Then by 6, $p < q$, $\tau^h(p) < \tau^h(q)$, $\tau^{2h}(p) < \tau^{2h}(q), \ldots, \tau^{(l-1)h}(p) < \tau^{(l-1)h}(q)$. By definition of q, this means that $p < \tau^h(p)$, $\tau^h(p) < \tau^{2h}(p)$, $\tau^{2h}(p) < \tau^{3h}(p), \ldots, \tau^{(l-1)h}(p) < \tau^{lh}(p) = p$, a contradiction.
8. Let P be a set of representatives of the orbits of τ. Then $\Phi(w)$ is the multiset $\{(w_p) \mid p \in P\}$. Therefore, by 7, the image of Φ is contained in **M**. Now, there are well-known length-preserving bijections between A^* and **M** (for example using factorization into Lyndon words), so the surjectivity of Φ follows.

3. The alternating lexicographic bijection

The *costandard permutation* of a word $w = a_1 \cdots a_n$ of length n on a totally ordered alphabet A is the permutation σ, denoted $cost(w)$, such that for any $i, j \in \{1, \ldots, n\}$, the condition $\sigma(i) < \sigma(j)$ is equivalent to $a_i < a_j$ or $a_i = a_j$ and $i > j$. The permutation $cost(w)$ may be obtained by numbering from right to left the letters of w, starting with the smallest letter, then the second smallest, and so on. For example,

b	a	a	c	b	c	a	b
6	3	2	8	5	7	1	4

so that $cost(baacbcab) = 63285714$.

Let ω be the longest permutation in S_n. Then it is easy to see that $cost(w) = st(w.\omega) \circ \omega$. It follows from the similar property for $st(w)$ that $cost(w)$ is the unique longest permutation $\sigma \in S_n$ such that $w.\sigma^{-1}$ is an increasing word.

We consider on infinite words the *alternating lexicographic order*: given two infinite words $a = a_0a_1a_2\cdots$ and $b = b_0b_1b_2\cdots$, we write $a <_{alt} b$ if either $a_0 < b_0$, or $a_0 = b_0$ and $a_1 > b_1$, or $a_0 = b_0$, $a_1 = b_1$ and $a_2 < b_2$, and so on. More formally, if for some i, one has $a_0 = b_0, \dots, a_{i-1} = b_{i-1}$ and $a_i < b_i$ if i is even and $a_i > b_i$ if i is odd.

We denote by $\mathbf{M}^-$ the set of multisets of necklaces on A described as follows: $M \in \mathbf{M}^-$ if and only if the necklaces in M are of the form (w) for some primitive word w, or of the form (ww) for some primitive word of odd length w; moreover, the necklaces of odd length appear at most once in M.

We define now a mapping Ξ, similar to the mapping of Section 2, but with the standard permutation replaced by the costandard permutation. It will turn out that the image of this mapping will be $\mathbf{M}^-$ (instead of $\mathbf{M}$), and that the inverse bijection may be computed by using the same kind of tableau as before, except that the lexicographic order must be replaced by the alternating lexicographic order.

Let $w = a_1 \cdots a_n$ be a word and τ the inverse of its costandard permutation. With each cycle $(j_1, \dots, j_i)$ of τ, associate the necklace $(a_{j_1}, \dots, a_{j_i})$. The mapping Ξ associates with w the multiset of necklaces obtained by taking all cycles of τ. Consider the example $w = baacbcab$ above. The cycles of $\tau = 63285714^{-1} = 73285164$ are $(1, 7, 6)$, $(2, 3)$, $(4, 8)$ and (5). The corresponding multiset of necklaces is therefore $\Xi(w) = \{(bac), (aa), (cb), (b)\}$.

In order to the define the inverse, we define the *alternating tableau* of a multiset of necklaces M: associate with M the multiset of words which is the union with multiplicities of the words in the conjugation classes appearing in M (there are i words, possibly repeated, for each circular word of length i, so that the cardinality of this multiset of words is equal to the total length of M). Now put in order, from top to bottom, these words in a right justified tableau, where the order is the alternating order of the infinite powers of these words; that is, u is above v in the tableau if $uuu\cdots <_{alt} vvv\cdots$. If the infinite powers are equal, the corresponding words are put in any order.

For example, consider the multiset $\{(bac), (aa), (cb), (b)\}$; its associated multiset of words is $\{bac, acb, cba, aa, aa, cb, bc, b\}$; these are ordered by

$$acb\cdots <_{alt} aa\cdots = aa\cdots <_{alt} bc\cdots <_{alt} bb\cdots <_{alt} bac\cdots <_{alt} cba\cdots <_{alt} cbcb\cdots.$$

Therefore, the alternating tableau is

a	c	b
	a	a
	a	a
	b	c
		b
b	a	c
c	b	a
	c	b

Theorem 3.1. *The previous mapping is a bijection between A^* and $\mathbf{M}^-$. The inverse mapping associates with a multiset the last column of its alternating tableau read from top to bottom.*

For later use, we consider the set $\mathbf{I}$ of independent sets of necklaces: a set E of necklaces is called *independent* if for any two circular words (u), (v) in E, the words u and v have no conjugate power.

Lemma 3.1. *There are canonical bijections between* $\mathbf{M}$, $\mathbf{M}^-$ *and* $\mathbf{I}$.

Proof. Note that necklaces are canonically in bijection with powers of Lyndon words. Using this identification, $\mathbf{M}$ is the set of finite multisets of Lyndon words; moreover, an element of $\mathbf{I}$ is a finite set of powers of distinct Lyndon words; we then obtain a bijection from $\mathbf{I}$ to $\mathbf{M}$ by replacing each power w^l of a Lyndon word w by the Lyndon word w with multiplicity l. Moreover, we obtain a bijection from $\mathbf{M}$ to $\mathbf{M}^-$ by replacing each Lyndon word of odd length, having multiplicity l, by w^2 with multiplicity l/2 times if l is even, and by w^2 with multiplicity $(l-1)/2$ times together with w with multiplicity 1, if l is odd. □

4. Proof of the main result

Recall from Section 3 that we denote by τ the inverse of the costandard permutation of $w = a_1 \cdots a_n$.

Lemma 4.1. *If $p < q$, then $a_{\tau(p)} \le a_{\tau(q)}$.*

Proof. The definition of *cost*(w) by numbering from right to left implies that the numbered positions are successively $\tau(1), \tau(2), \ldots$ and so on. This implies the lemma. □

Lemma 4.2. *If $p < q$ and if $a_{\tau(p)} = a_{\tau(q)}$, then $\tau(p) > \tau(q)$.*

Proof. The definition of *cost*(w) by numbering from right to left shows that if $i < j$ and $a_i = a_j$, then $cost(i) > cost(j)$. Suppose now that $p < q$ and $a_{\tau(p)} = a_{\tau(q)}$. Put $i = \tau(p)$ and $j = \tau(q)$; then $a_i = a_j$; if we had $\tau(p) < \tau(q)$, that is, $i < j$, we would have $cost(i) > cost(j)$, that is, $p > q$, a contradiction. Thus $\tau(p) > \tau(q)$. □

For $p = 1, \ldots, n$, let $w_p = a_{\tau(p)} a_{\tau^2(p)} \ldots a_{\tau^k(p)}$ where k is the length of the cycle of τ containing p. Note that therefore $\tau^k(p) = p$ and the last letter of w_p is a_p. Let P be a set of representatives of the orbits of τ. Then observe that $\Xi(w)$ is the multiset $\{(w_p) \mid p \in P\}$.

Corollary 4.1. *If $p < q$ then $w_p^\infty \le_{alt} w_q^\infty$.*

Proof. Since $w_p^\infty = a_{\tau(p)} a_{\tau^2(p)} a_{\tau^3(p)} \ldots$, the corollary immediately follows from the two lemmas. □

We may deduce the injectivity of the mapping Ξ from the corollary. By definition of Ξ, the multiset of words which is the union with multiplicities of the words in the conjugation classes appearing in Ξ is exactly the multiset of words $w_p, p = 1, \ldots, n$. Thus, the alternating tableau (as defined in Section 3) of $\Xi(w)$ is by the corollary equal to the right justified tableau whose p-th row is the word w_p. Since the last letter of that word is a_p, the injectivity follows, together with the construction of the inverse through the alternating tableau.

Lemma 4.3. *If $p < q$ and $w_p = w_q$, then $\tau^i(p) < \tau^i(q)$ if i is even and $\tau^i(p) > \tau^i(q)$ if i is odd.*

Proof. Note that if $w_p = w_q$, then $w_{\tau^i(p)} = w_{\tau^i(q)}$. If moreover $p < q$, then by Lemma 4.2, $\tau(p) > \tau(q)$ (since $w_p = w_q$ implies that their first letters coincide, that is, $a_{\tau(p)} = a_{\tau(q)}$). The lemma follows by induction on i. □

Corollary 4.2. *Let $w_p = u^l$ for some word u and some integer $l \ge 1$.*

(i) *If the length of u and l are both odd, then $l = 1$.*
(ii) *If the length of u is even, then $l = 1$.*

Proof. Let h be the length of u and $q = \tau^h(p)$. We have therefore $w_p = w_q$. Note that hl is the length of w_p, that is the length of the cycle of τ at p. We assume by contradiction that $l > 1$, so that p and q are not equal. We assume below that $p < q$, since the arguments are quite similar for $p > q$.

(i) Suppose that h and l are both odd. Then, since hl is odd, we have by Lemma 4.3, $\tau^{hl}(p) > \tau^{hl}(q)$. Now, $\tau^{hl}(p) = p$. Moreover, $\tau^{hl}(q) = \tau^{hl}(\tau^h(p)) = \tau^h(p) = q$. Thus, $p > q$, a contradiction.
(ii) Suppose that h is even. Then by Lemma 4.3, $p < q$, $\tau^h(p) < \tau^h(q)$, $\tau^{2h}(p) < \tau^{2h}(q), \ldots, \tau^{(l-1)h}(p) < \tau^{(l-1)h}(q)$. That is, $p < \tau^h(p)$, $\tau^h(p) < \tau^{2h}(p)$, $\tau^{2h}(p) < \tau^{3h}(p), \ldots, \tau^{(l-1)h}(p) < \tau^{lh}(p) = p$, a contradiction. □

We may show now that the image of Ξ is contained in the set $\mathbf{M}^-$. It is enough to show that each word w_p is either primitive or the square of a primitive word of odd length, and that if $w_p = w_q$ is of odd length, then $p = q$.

The corollary shows that if w_p is not primitive, then $w_p = u^l$ with u of odd length and l even; if $l \geq 4$, then $l = 2m, m \geq 2$ and $w_p = v^m$, where $v = u^2$ is of even length, which contradicts the corollary; hence $l = 2$.

Suppose that $w_p = w_q$ is of odd length k (which is the length of the cycle of τ at p and q); if $p < q$, then by Lemma 4.3, we have $\tau^k(p) > \tau^k(q)$, that is, $p > q$, a contradiction. Thus we must have $p = q$.

All this shows that the image of Ξ is contained in $\mathbf{M}^-$. Now, there exist length-preserving bijections between A^* and $\mathbf{M}^-$: for example using factorization into Lyndon words and using Lemma 3.1. Thus the surjectivity of Ξ follows.

5. Applications to symmetric functions and representations of the symmetric group

5.1. With the lexicographical bijection

We recall first the construction and the properties of the symmetric functions related to the bijection Φ of [4].

Given a multiset M of necklaces (or circular words) of total length n, we associate with it a partition of n, denoted $\lambda(M)$ and called the *cycle type* of M: the parts of $\lambda(M)$ are the lengths of the necklaces forming M, with multiplicities. The same construction applies to permutations in S_n, viewed as multisets (actually sets) of their cycles.

Let A be an infinite alphabet, which will be considered as a set of noncommuting and of commuting variables, depending on the context. Given a word $a_1 \dots a_n$ and the corresponding necklace $(a_1 \dots a_n)$ on A, the *evaluation* of both of them is the commutative monomial $a_1 \dots a_n$ in $\mathbb{Z}[A]$. The evaluation of a multiset of primitive necklaces M is the monomial in $\mathbb{Z}[A]$ which is the product of the evaluations of all necklaces that form M.

Fix a partition λ. In [4] (see also [11, Theorem 9.41]), the following element of $\mathbb{Z}[[A]]$ is constructed:

$$P_\lambda = \sum_{\lambda(M)=\lambda} ev(M),$$

where the sum is over all multisets of primitive necklaces on A of cycle type λ. Since bijections from A into A preserve primitive necklaces, it is easy to see that P_λ is a symmetric function. Note that for a partition with a single part n, this symmetric function P_n is simply the sum of the evaluations of all primitive necklaces of length n; equivalently, the sum of all the evaluations of the Lyndon words of length n.

It follows immediately from the bijectivity of Φ that one has also

$$P_\lambda = \sum_{\lambda(st(w))=\lambda} ev(w),$$

where the sum is over all words on A whose standard permutation has cycle type λ.

This symmetric function may be described as follows. Let $\lambda = 1^{n_1} \dots k^{n_k}$: this means that the part i has multiplicity n_i in λ. Then by the definition of P_λ one sees that

$$P_\lambda = \prod_{1 \leq i \leq k} P_{i^{n_i}}.$$

Now, by the combinatorial definition of plethysm (see [5, Section I.8]), one sees that

$$P_{i^{n_i}} = h_{n_i}[P_i],$$

where h_n denotes the n-th homogeneous symmetric functions, with the notations of [5]. Thus we have

$$P_\lambda = \prod_{1 \leq i \leq k} h_{n_i}[P_i].$$

These symmetric functions correspond to representations and characters of the symmetric groups as is described in [4,11]. We briefly review the construction. Consider the algebra of noncommutative polynomials $\mathbb{Q}\langle A \rangle$ on the infinite set of noncommuting variables $a \in A$. The *free Lie algebra* is the smallest subspace $\mathbf{L}$ of $\mathbb{Q}\langle A\rangle$ containing the variables in A and closed under the Lie bracket $[f,g] = fg - gf$. Denote by $\mathbf{L}_n$ the n-th homogeneous part of $\mathbf{L}$. Denote

$$(f_1,\ldots,f_l) = \frac{1}{l!}\sum_{\sigma\in S_l} f_{\sigma(1)}\cdots f_{\sigma(l)}.$$

Now for any partition λ, denote U_λ the subspace of $\mathbb{Q}\langle A\rangle$ spanned by the elements $(f_1,\ldots,f_l)$, for all $P_i \in \mathbf{L}_{\lambda_i}$. Then it is known (and follows from the Poincaré–Birkhoff–Witt theorem) that

$$\mathbb{Q}\langle A\rangle = \bigoplus_\lambda U_\lambda,$$

where the sum is over all partitions λ. Then P_λ is the (Frobenius) character (or characteristic map as in [5]) of the representation of the symmetric group S_n acting on the multilinear part of degree n of U_λ; equivalently, P_λ is the multivariate generating function of U_λ. In particular, P_n is the character of the n-th Lie representation. For later use, we recall the formula giving P_n:

$$P_n = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}},$$

where μ is the Möbius function and p_d the power sum symmetric function.

5.2. With the alternating bijection

We mimic now the previous construction by replacing the multisets and the bijection.

Recall that we denote by $\mathbf{M}^-$ the set of multisets of necklaces on A described as follows: $M \in \mathbf{M}^-$ if and only if the necklaces in M are of the form (w) for some primitive word w, or of the form (ww) for some primitive word of odd length w; moreover, the necklaces of odd length appear at most once in M.

Fix a partition λ. Then define

$$Q_\lambda = \sum_{M\in\mathbf{M}^-,\ \lambda(M)=\lambda} ev(M) \in \mathbb{Z}[[A]].$$

For the same reason as before, this is a symmetric function. By our bijection between $\mathbf{M}^-$ and words, we have also

$$Q_\lambda = \sum_{w\in A^*,\ \lambda(cost(w))=\lambda} ev(w).$$

We may as before describe the symmetric function as follows. Denote by e_n the n-th elementary symmetric function. Then the definition of $\mathbf{M}^-$ implies that for $\lambda = 1^{n_1}\ldots k^{n_k}$, we have

$$Q_\lambda = \prod_{1\le i\le k,\, i\, even} h_{n_i}[Q_i] \prod_{1\le i\le k,\, i\, odd} e_{n_i}[Q_i]. \tag{1}$$

Recall from [5] that there is a fundamental involution on the ring of symmetric functions, denoted by ω.

Lemma 5.1. *If n is twice an odd number, then $\omega(P_n)$ is equal to the sum of the evaluations of the Lyndon words of length n together with the square of Lyndon words of length $n/2$.*

Proof. The sum of the evaluations of the Lyndon words of length n is equal to P_n as noted above. By definition of plethysm, the sum of the evaluations of the squares of the Lyndon words of length m is equal to $P_m[p_2]$; indeed this sum is equal to the sum of the evaluations of the words obtained from Lyndon words of length m by doubling each letter in it. Thus, putting $n = 2m$, we have that the sum of the lemma is equal to

$$P_n + P_m[p_2] = \frac{1}{n}\sum_{d|n}\mu(d)p_d^{\frac{n}{d}} + \frac{1}{m}\sum_{d|m}\mu(d)p_d^{\frac{m}{d}}[p_2].$$

Now, suppose that m is odd. Since $p_d[p_2] = p_{2d}$, the last sum is equal to

$$\frac{1}{n}\sum_{d|n, d\, odd} \mu(d) p_d^{\frac{n}{d}} + \frac{1}{n}\sum_{d|n, d\, even} \mu(d) p_d^{\frac{n}{d}} + \frac{1}{m}\sum_{d|m} \mu(d) p_d^{\frac{m}{d}}[p_2]$$
$$= \frac{1}{n}\sum_{d|n, d\, odd} \mu(d) p_d^{\frac{n}{d}} + \frac{1}{n}\left(\sum_{d|m} \mu(2d) p_{2d}^{\frac{m}{d}} + 2\sum_{d|m} \mu(d) p_{2d}^{\frac{m}{d}}\right).$$

Since for d odd, $\mu(2d) = -\mu(d)$, this is equal to

$$\frac{1}{n}\left(\sum_{d|n, d\, odd} \mu(d) p_d^{\frac{n}{d}} + \sum_{d|m} \mu(d) p_{2d}^{\frac{m}{d}}\right).$$

On the other hand, since $\omega(p_d) = (-1)^{d-1} p_d$, we have

$$\omega(P_n) = \frac{1}{n}\sum_{d|n} \mu(d)(-1)^{(d-1)\frac{n}{d}} p_d^{\frac{n}{d}}.$$

Now $(d-1)\frac{n}{d} = n - \frac{n}{d}$ and n is even, so that the last sum is equal to

$$\frac{1}{n}\left(\sum_{d|n, d\, odd} \mu(d) p_d^{\frac{n}{d}} - \sum_{d|n, d\, even} \mu(d) p_d^{\frac{n}{d}}\right) = \frac{1}{n}\left(\sum_{d|n, d\, odd} \mu(d) p_d^{\frac{n}{d}} - \sum_{d|m} \mu(2d) p_{2d}^{\frac{m}{d}}\right).$$

This concludes the proof. □

Lemma 5.2 *([4, p. 201]). If n is not twice an odd number, then $\omega(P_n) = P_n$.*

We deduce the following result.

Corollary 5.1. *$Q_\lambda = \omega(P_\lambda)$.*

Proof. We first show that $\omega(P_n) = Q_n$. If n is not twice an odd number, in view of Lemma 5.2, this amounts to show that $P_n = Q_n$. Now P_n is the sum of the evaluations of the primitive necklaces of length n, and Q_n is the sum of the evaluations of the necklaces in $\mathbf{M}^-$ of length n; these necklaces are primitive because of the definition of $\mathbf{M}^-$ and the hypothesis on n. Hence P_n and Q_n are equal.

Now suppose that n is twice an odd number. Then in view of Lemma 5.1, we have to show that Q_n is the sum of the evaluations of the Lyndon words of length n and of the square of Lyndon words of length $n/2$. Now this follows from the definition of $\mathbf{M}^-$, since Q_n is the sum of the evaluations of the necklaces of length n in $\mathbf{M}^-$.

By the formula above giving P_λ and Q_λ, and since ω is an automorphism, in order to prove the lemma, it is enough to show that $\omega(h_n[P_i]) = h_n[Q_i]$ if i is even, and $= e_n[Q_i]$ is i is odd. Now, for homogeneous symmetric functions f, g with g of degree i, it is well-known that $\omega(f[g]) = f[\omega(g)]$ if i is even, and $= \omega(f)[\omega(g)]$ if i is odd. Hence the corollary follows from the equality $\omega(h_n) = e_n$ and $\omega(P_i) = Q_i$. □

Remark. There is an alternative proof of the corollary that works on the other side of the bijections. Since it is of some interest, we sketch it now. It rests on the fact that if a multiset of monomials is determined by a family of weak and strict inequalities on the variables, and if the sum of this multiset is a symmetric function f, then $\omega(f)$ is obtained by interchanging weak and strict inequalities; see [6, Theorem 3.1]. We show on an example that this result implies the corollary. Indeed, consider the permutation $\sigma = 5371624 \in S_n$ and a word w of length 7, that we may write as $w = a_5a_3a_7a_1a_6a_2a_4$. Then $st(w) = \sigma$ if an only if $a_1 \le a_2 < a_3 \le a_4 < a_5 \le a_6 < a_7$. Moreover, $cost(w) = \sigma$ if and only if $a_1 < a_2 \le a_3 < a_4 \le a_5 < a_6 \le a_7$.

We recall now some facts on Lie superalgebras which will allow us to describe the representation which has the character Q_λ. Consider again $\mathbb{Q}\langle A\rangle$. The *free oddly generated Lie superalgebra* is the smallest subspace $\mathbf{L}^-$ of $\mathbb{Q}\langle A\rangle$ containing the variables in A and closed under the following operation (f, g are homogeneous polynomials): $[f,g]^- := fg - (-1)^{deg(f)deg(g)}gf$. Denote by $\mathbf{L}_n^-$ the n-th homogeneous part of $\mathbf{L}^-$. Denote also, for homogeneous polynomials $f_1, \ldots, f_l$,

$$(f_1, \ldots, f_l)^- := \frac{1}{l!}\sum_{\sigma\in S_l}(-1)^{\sum_{i<j,\sigma(i)>\sigma(j)} deg(f_i)deg(f_j)} f_{\sigma(1)}\cdots f_{\sigma(l)}.$$

Then the following lemma follows from the "super" version of the Poincaré–Birkhoff–Witt theorem (see [10]).

Lemma 5.3.

$$\mathbb{Q}\langle A\rangle = \bigoplus_\lambda U_\lambda^-,$$

where the sum is over all partitions λ and where, for $\lambda = \lambda_1\cdots\lambda_l$, U_λ is spanned by the elements $(f_1, \ldots, f_l)^-$, for all $f_i \in \mathbf{L}_{\lambda_i}^-$.

We give a proof for the reader's convenience.

Proof. We claim that for any homogeneous polynomials $f_1, \ldots, f_l$ and any permutation σ, one has

$$f_1\cdots f_l \equiv (-1)^{\sum_{i<j,\sigma(i)>\sigma(j)} deg(f_i)deg(f_j)} f_{\sigma(1)}\cdots f_{\sigma(l)} \bmod (\mathbf{L}^-)^{\le l-1},$$

where the last symbol denotes the space spanned by products of no more than $n-1$ elements of $\mathbf{L}^-$. The claim will be proved below.

It implies that

$$f_1\ldots f_l \equiv (f_1, \ldots, f_l)^- \bmod (\mathbf{L}^-)^{\le l-1}.$$

Now let $g_1, g_2, \ldots$ denote a homogeneous basis of $\oplus_{n\, even} L_n^-$ and $h_1, h_2, \ldots$ denote a homogeneous basis of $\oplus_{n\, odd} L_n^-$. Order the g_i's and the h_j's naturally, with the former before the latter. Then by [10, Theorem 3.1] (a version of the Poincaré–Birkhoff–Witt theorem), the set of all products in weakly increasing order of the g_i and h_j's, with the restriction that a h_j can appear at most once, forms a basis of $\mathbb{Q}\langle A\rangle$. The previous equation implies therefore that if each product $f_1\ldots f_l$ is replaced by $(f_1, \ldots, f_l)^-$, then this new set is also a basis. Therefore, since the latter operator is l-multilinear, the set of these elements with λ equal to the multiset $\{deg(f_1), \ldots, deg(f_l)\}$ (considering partitions as multiset of integers) is a basis of U_λ^-. Thus the lemma follows.

It remains to prove the claim. We do it by induction on the length (number of inversions) of σ. If it is 0, there is nothing to prove. Otherwise we may write $\sigma = \alpha\circ\tau$, with τ the adjacent transposition $(k, k+1)$ and α having one less inversion than σ. Write $i(\sigma) = \sum_{i<j,\sigma(i)>\sigma(j)} deg(f_i)deg(f_j)$. Then $i(\sigma) = i(\alpha) + deg(f_{\alpha(k)})deg(f_{\alpha(k+1)})$ since $\sigma(1)\ldots\sigma(l) = \alpha(1)\cdots\alpha(k-1)\alpha(k+1)\alpha(k)\alpha(k+2)\cdots\alpha(l)$ and $\alpha(k) < \alpha(k+1)$ by assumption on the inversions.

We have

$$f_{\alpha(k)}f_{\alpha(k+1)} = [f_{\alpha(k)}, f_{\alpha(k+1)}]^- + (-1)^{deg(f_{\alpha(k)})deg(f_{\alpha(k+1)})} f_{\alpha(k+1)}f_{\alpha(k)}.$$

Thus, by multiplying appropriately on the left and on the right, we obtain

$$f_{\alpha(1)}\ldots f_{\alpha(l)} \equiv (-1)^{deg(f_{\alpha(k)})deg(f_{\alpha(k+1)})} f_{\sigma(1)}\ldots f_{\sigma(l)}.$$

Now by induction, $f_1\ldots f_l \equiv (-1)^{i(\alpha)} f_{\alpha(1)}\ldots f_{\alpha(l)}$. Thus we obtain that $f_1\ldots f_l \equiv (-1)^{i(\sigma)} f_{\sigma(1)}\ldots f_{\sigma(l)}$, as desired. □

Theorem 5.1. *Let λ be a partition of n. Then Q_λ is the character of the symmetric group S_n acting on the multilinear part of U_λ^-. Equivalently, it is the multivariate generating function of U_λ^-.*

Proof. It suffices to prove the second assertion. We consider first the case of a partition λ with only one part n. We have to find a finely homogeneous (that is, homogeneous with respect to each variable) basis of $U_\lambda^- = \mathbf{L}_n^-$ whose multivariate generating function is $Q_\lambda = Q_n$. This will follow from the theory of Lyndon bases, or more generally of Hall bases, adapted to the Lie superalgebra case; see [9], [11, Section 4.4.4]. Indeed, a basis of $\mathbf{L}_n^-$ is obtained as follows. Take a set of Hall trees and evaluate it in $\mathbf{L}^-$ by interpreting the binary operation as the bracketing $[\,,\,]^-$. Then $\mathbf{L}^-$ has as basis the set of all polynomials f obtained in this way, together the polynomials $[f, f]^-$ with f of odd degree. Since the multivariate generating function of the Hall trees (or equivalently Lyndon words) of degree n is classically equal to P_n (see e.g. [11, Theorem 7.2]), we see by Lemmas 5.1 and 5.2 that $\mathbf{L}_n^-$ has a multi-homogeneous basis whose multivariate generating function is $\omega(P_n)$, equal to Q_n by Corollary 5.1.

Consider now a general partition λ. The proof of Lemma 5.3 shows that U_λ^- has the following basis: the set B_λ of $(f_1, \ldots, f_l)^-$, where $f_1, \ldots, f_l$ is a weakly increasing sequence of elements in the set $B = \{g_1, g_2, \ldots, h_1, h_2, \ldots\}$. Here the f_i's and the g_j's are as in the proof of this lemma, and we may even assume that they are finely homogeneous. Then the basis B_λ that we obtain is also finely homogeneous. It follows, by Eq. (1) and the definition of plethysm, that its multivariate generating function is Q_λ. $\square$

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