International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

Michel Fliess and non-commutative formal power series

C. Reutenauer [a]

[a] Déépartement de mathéématiques, Universitéé du Quéébec Montrééal, Case postale 8888, succursale Centre-Ville, Montrééal (Quéébec), Canada

Available online: 06 Oct 2010

International Journal of Control
Vol. 81, No. 3, March 2008, 338–343

Michel Fliess and non-commutative formal power series†

C. REUTENAUER*

Département de mathématiques, Université du Québec à Montréal, Case postale 8888, succursale Centre-Ville, Montréal (Québec) Canada

(*Received 4 July 2007; in final form 5 July 2007*)

We give an overview of the work of Michel Fliess on the applications of non-commutative formal power series in theoretical computer science.

1. Introduction

I shall give here an overview of the first researches of Michel Fliess. They correspond to about 15 articles; the first, by Fliess and Richard (1969), on the Hurwitz product, now called shuffle product, and the last by Fliess (1975) on positive rational series. One has to add to these articles his PhD in 1972 entitled: Sur certaines familles de séries formelles. The corresponding work is on the algebraic and arithmetic aspects of non-commutative formal power series, together with their applications in theoretical computer science, mostly formal language theory.

I shall not mention his work on control theory, nor my collaboration with him in the early 80s (applications of differential Galois theory, the so called Picard–Vessiot theory, to bilinear systems). Let me however observe that his first article in control theory appears already in Fliess (1973), an application of series to bilinear systems.

Help of the three referees is gratefully acknowledged.

2. Automata and Kleene's theorem

An automaton is a finite directed graph whose edges are labelled in some finite set, called the alphabet; there is also an initial state, and a set of final states (the word state is preferred here to the word vertex). See figure 1, where the initial state is (resp. the final states are) represented by an arrow pointing in (resp. pointing out).

By reading the labels on the consecutive edges of a path, one obtains a word on the alphabet. If one considers all possible paths form the initial state to some final state, one obtains a set of words: it is the language recognized by the given automaton. In figure 1, this language contains *a*, *aba*, *ac*, *acaba* and many other words.

Consider figure 2. The language of this automaton contains all words *abab*...*ab*, together with all words obtained from the previous ones by putting *c*'s between a *b* and an *a*, for example *cabccabcccabab*. A useful notation for the latter language is $\{ab, c\}^*$: the exponent * means the submonoid (of the free monoid generated by the alphabet) generated by *ab* and *c*; the latter is the set of words obtained by concatenating in all possible ways the words *ab* and *c*. The notation L^* is called the Kleene star-operation and is, algebraically speaking, the submonoid generated by L.

Theorem 1 (Kleene 1956): *A language is recognized by some finite automaton if and only if it may be obtained from finite languages by the operations union, concatenation and star.*

Note that the concatenation of two languages L and L' is naturally defined as $\{ww' | w \in L, w' \in L'\}$, where ww' means the concatenation of the two words w and w' (a highly non-commutative operation).

*Email: christo@math.uqam.ca
†Dedicated to Michel Fliess for his 60th birthday.

International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online © 2008 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170701556898

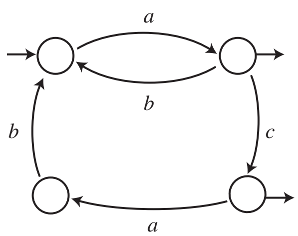

Figure 1. An automaton.

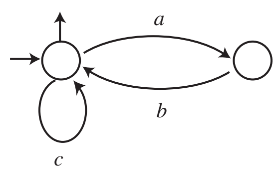

Figure 2. An automaton.

As an example illustrating Kleene's theorem, consider the language of words on the alphabet $\{a, b\}$ having an even number of b's. It has the expression

$$(a \cup ba^* b)^*$$

and is recognized by the automaton of figure 3.

3. Schützenberger and non-commutative rationality

A basic discovery of Schützenberger, concerning Kleene's theorem, is that the operations involved in it are rational operations, and that automata correspond to linear representations (of the free monoid). The word rational used here has the following meaning: regarding usual numbers or polynomials, the algebraic operations are sum, product and subtraction, whereas rational operations include also division. Thus rational numbers are those obtained from natural integers using these 4 operations; similarly for rational fractions.

In the Kleene theorem, revisited by Schützenberger, addition becomes union, product becomes concatenation and division is replaced by the star operation. The first two analogies are self-explanatory, while the third one is illustrated by the following example, with $L = \{ab, c\}^*$, and when one computes in the ring of series in the non-commuting variables a, b, c : $\sum_{w \in L} w = \sum_{n \geq 0} (ab + c)^n = (1 - ab - c)^{-1}$. Thus, by means of a well-known object in mathematics, the geometric series, one is led to relate inversion and submonoid generation; in the example, the (characteristic series of the) submonoid generated by ab and c is the inverse of $1 - ab - c$.

Regarding recognizability by automata and representations, the relation between them is made through the incidence matrices, a well-known principle in graph theory. For example, in figure 2, if we look only at the edges having a given label, we obtain three graphs, whose incidence matrices are respectively

$$\mu(a) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mu(c) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The mapping μ extends uniquely into a homomorphism from the free monoid generated by a, b, c into the multiplicative monoid of 2 by 2 matrices; for any word w, $\mu(w)_{i,j}$ is equal to the number of paths from i to j labelled w, as follows from a straightforward induction on the length of w.

Generalizing all this, we consider the semiring $K\langle\langle A\rangle\rangle$ of non-commutative series in the non-commuting variables in A over the semiring K. This semiring has an extra operation *, which is $S^* = \sum_{n \geq 0} S^n$, defined when S has no constant term (if K is a ring, $S^* = (1 - S)^{-1}$).

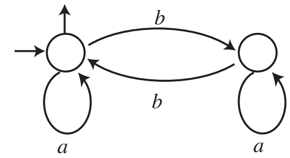

Figure 3. Words with an even number of b's.

Theorem 2 (Schützenberger 1961): *A series is rational (that is, obtainable from polynomials by semiring operations, together with the star operation) if and only if it is some component of a linear representation over K of the free monoid.*

The final condition means that for some n, some homomorphism μ from the free monoid generated by A into the multiplicative monoid of $n \times n$ matrices over K, and some $i, j \in \{1, \ldots, n\}$, the series is $\sum_w (\mu w)_{i,j} w$. This theorem justifies to call rational the Kleene languages.

Schützenberger's theorem is a non-commutative generalization of a classical result: a sequence (a_n) satisfies a linear recursion if and only if the series $\sum_{n \geq 0} a_n x^n$ is the quotient of two polynomials. For example, for the Fibonacci sequence (F_n) defined by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, one has $\sum_{n \geq 0} F_n x^n = (1/1 - x - x^2)$.

For rational non-commutative series, the reader may see the books by Eilenberg (1974), Salomaa and Soittola (1978) and Berstel and Reutenauer (1988).

4. Here comes Fliess

I don't know how Michel Fliess came in contact with Schützenberger, but I suppose that this happened in the late 60s. I suppose that, among several subjects in theoretical computer science, he was mostly attracted by the algebraic aspects and thus, by non-commutative formal power series, which generalize languages by introducing more algebraic structure (actually, it was this that happened to me years later, when I read the book of Gross and Lentin on formal grammars).

In any case, he became one of the rare direct students of Schützenberger, who had a dozen of them end 60s to beginning 70s.

Regarding their applications to theoretical science, and their algebraic and arithmetical aspects, the work of Fliess on non-commutative formal power series includes:

- algebraic series;
- transductions;
- Hadamard and Hurwitz (shuffle) product;
- Cohn's free field;
- rational subsets of free groups;
- stochastic languages;
- Hankel matrices;
- positive rational series;
- centralizers,

and many others. I shall illustrate below only few of these subjects.

5. Algebraic series and the Chomsky hierarchy

This hierarchy is a filtration in 4 parts of the recursively enumerable languages; the latter are the languages that are algorithmically enumerable. The filtration is: $\{$languages of finite automata$\} \subset \{$context-free languages$\} \subset \{$context-sensitive languages$\} \subset \{$recursively enumerable languages$\}$.

A context-free language is a language that can be generated by a context-free grammar, or Chomsky grammar. Such a grammar has rules that mimick and formalize the following rule of usual grammar: Sentence = subject + verb + complement. An example of such formalized grammar is: $S \to aSS \mid b$, which generates the so called Lukasiewicz language $\{a, abb, ababb, \ldots\}$; for example, the word $ababb$ is generated by the derivation $S \to aSS \to abS \to abaSS \to ababS \to ababb$; see Berstel (1978).

A striking observation of Schützenberger is that this generation by derivations correspond to solving non-commutative algebraic equations. In the example above, one sees that the series S, which is the sum of all words in the Lukasiewicz language, is the unique solution of the equation $S = aSS + b$. That's why Schützenberger called algebraic the context-free languages, and corresponding series.

Thus there is a close analogy between the two first terms of the Chomsky hierarchy and the inclusion $\{$rational numbers$\} \subset \{$algebraic numbers$\}$, and even more closely with the inclusion $\{$rational functions$\} \subset \{$algebraic functions$\}$. It is this latter analogy which was studied closely by Fliess. There exists indeed the notion of algebraic series in one variable: these are the series (over $\mathbb{C}$ say) $S = \sum_{n \geq 0} a_n x^n$ which are solution of an algebraic equation $P_0 S^d + P_1 S^{d-1} + \cdots + P_d = 0$, where the P_i's are polynomials in x (not all zero). Of course, this extends to series the familiar notion of algebraic number (an algebraic number is a complex number s such that $p_0 s^d + p_1 s^{d-1} + \cdots + p_d = 0$ for some integers p_i, not all zero). He defined the notion of algebraically constructive series; roughly speaking, such a series satisfies a nice equation, in the sense that it is the unique series which is a solution of this equation (other solutions may, and must, exist, but are not formal power series). Using a deep theorem of Fürstenberg, he showed that the algebraic series in one variable, in the sense of Schützenberger, coincide with the algebraically constructive series. This clarifies considerably the definitions, Fliess (1971).

6. Positive rational series

Here is one of my favourite results of Fliess. The general setting is the study of noncommutative series over semirings. A semiring is, roughly speaking, a ring without subtraction; for example, $\mathbb{N}, \mathbb{Q}_+, \mathbb{R}_+$ and all rings of course. In the Schützenberger theory of series, semirings were necessary in order to have as special case the languages, which correspond to the boolean semiring $\mathbb{B} = \{0, 1\}$ with $1 + 1 = 1$; indeed, to a language L corresponds bijectively the series $\sum_{w \in L} w \in \mathbb{B}\langle\langle A \rangle\rangle$.

Meanwhile, semirings have become more familiar to other mathematicians: after automata theory, in control theory appeared the so-called tropical semiring $(\mathbb{R}_+, \max, +)$ and its variants; there exists even now tropical combinatorics and the fashionable tropical algebraic geometry.

Let's go back to series. A natural problem is the following: let $S \in \mathbb{N}\langle\langle A \rangle\rangle$ be a rational series over $\mathbb{Q}_+$; this means that S has coefficients in $\mathbb{N}$ and has a rational expression involving positive rational numbers. Can one conclude that S is rational over $\mathbb{N}$, that is, that S has a also rational expression involving only coefficients in $\mathbb{N}$? Fliess showed that the answer is yes Fliess (1975). This result appears to be somewhat isolated, in the sense that many problems which are analogue (the so called Fatou extension problems) have a negative answer. I could find for example a series which is rational over $\mathbb{R}_+$, has coefficients in $\mathbb{Q}_+$, and is not rational over $\mathbb{Q}_+$. I remember that when I called Michel to tell him this discovery, he said "Dieu n'existe pas".

Not only is the result of Fliess quoted above isolated (hence has some mystery) and extremely elegant, but his proof also is: it uses some nice results of the Eilenberg-Schützenberger theory of rational subsets in commutative monoids.

Let me finally mention that the Fatou problems alluded above are a generalization of the following result of Fatou: if a sequence (a_n) of integers satisfies a linear recursion: $\forall n, a_{n+k} = \alpha_1 a_{n+k-1} + \alpha_2 a_{n+k-2} + \cdots + \alpha_k a_n$, and if k is chosen minimum, then the α_i's are integers.

7. The Hankel matrix

This work of Fliess has had much influence, especially in Control Theory (bilinear systems). Let me recall first the classical case. Consider a linearly recurrent sequence $(a_n)_{n\geq 0}$ and form the infinite matrix $(a_{i+j})_{i,j\geq 0}$, called the Hankel matrix. Then it is well-known that this matrix has finite rank (the rank of an infinite matrix is defined as in the finite case). Conversely, if the rank of this matrix is finite, then the sequence is linearly recurrent; and this rank is equal to the length of the shortest linear recursion satisfied by the sequence.

Fliess (1974a) extends all this to noncommutative series. Let $S = \sum_w (S, w)w$ be such a series, where the sum is over all words w in some finitely generated free monoid X^*. Then its Hankel matrix is the infinite matrix, with rows and columns indexed by X^*, defined by $H_S = ((S, uv))_{u, v \in X^*}$. Then he shows that S is rational if and only if H_S is of finite rank. He uses these methods to clarify some results of Schützenberger. In particular, he introduces a class of representations of S, wider than those considered by the latter (see § 3); this allows him first to show that the dimension of the representations of minimal dimension, for a given series S, is equal to the rank of H_S; then, to show that all minimal representations are conjugate, an important theoretical result, also useful for practical computations; and also, to make the fundamental observation that there is a natural duality between the space spanned by the rows and the columns of the Hankel matrix. Note that his new class of minimal representations helped me to find a similar class of minimal representations for the elements of the free field: these elements are a kind of non-commutative fractions, and the minimal representation is a kind of reduced fraction. In this minimal representation, the matrix to be inverted is linear, instead of being polynomial of any degree; see Cohn and Reutenauer (1999). Furthermore, the minimal representation allowed me to show that the inversion height is unbounded; see Reutenauer (1996).

All this has a flavour of Hopf algebra theory, and indeed there are precise links. Recall that when $\mathcal{A}$ is an algebra over a field K, then the dual $\mathcal{A}^*$ of $\mathcal{A}$, viewed as vector space, is naturally a left and right module over $\mathcal{A}$. Then an element f of $\mathcal{A}^*$ is called a representative function on $\mathcal{A}$ if $\mathcal{A} \cdot f$ (or equivalently $f \cdot \mathcal{A}$) is of finite dimension over K. The set of representative functions is called the Sweedler dual of $\mathcal{A}$. It is also the set of components of linear representations of $\mathcal{A}$, see Sweedler (1969).

Thus, since the space of series $K\langle\langle A\rangle\rangle$ is naturally the dual of the space of polynomials $K\langle A\rangle$, the set of recognizable series appears as the Sweedler dual of $K\langle A\rangle$. A recognizable series is the same thing as a representative function on $K\langle A\rangle$.

8. Hopf algebras and shuffle

While the closure under the usual noncommutative product (or Cauchy product) of rational series is evident from the definitions, their closure under Hadamard product follows from Schützenberger's theorem, by taking the tensor product of the corresponding representations. Fliess noted that a similar construction gave the closure under shuffle product, and gave also several generalizations, Fliess (1969, 1974b). The shuffle product has a lot of applications in control theory; in particular, the product of the outputs of two bilinear systems corresponds to the shuffle of their Fliess series, which are rational. These results originate in Chen's work (1957), revisited by Ree (1958).

With the shuffle product, one has another link with Hopf algebras. I remember that in the 70s, Michel told me: "Hopf algebras are fundamental; they will be taught as group theory today". This was 10 years before the invasion of quantum groups in mathematics.

Let me give some precision on the link with Hopf algebras. The K-algebra of non-commutative polynomials $K\langle A\rangle$ (or the tensor algebra) appears to be the enveloping algebra of its Lie subalgebra generated by the variables in A (this Lie algebra is freely generated by them); as each enveloping algebra, it has a natural coproduct defined by $a \mapsto a \otimes 1 + 1 \otimes a$. Now, the dual space of $K\langle A\rangle$ is, as already observed, the space $K\langle\langle A\rangle\rangle$of series; it has therefore a natural product, adjoint to the previous coproduct: it turns out that this product is the shuffle product. Then $K\langle A\rangle$, with its noncommutative (Cauchy or concatenation) product and the coproduct above, is a Hopf algebra, and dually (since $K\langle A\rangle$ is the graded dual of itself), so is $K\langle A\rangle$ with the shuffle product and a coproduct Δ adjoint to the Cauchy product. Note that the latter is true also for the shuffle algebra of rational series, since rational series S are characterized by the fact that $\Delta(S) = \sum_{\text{finite}} S_1 \otimes S_2$, see Reutenauer (1993, p. 38).

Shuffle product and the previous duality allows to lay down a non-commutative umbral calculus, extending Rota's in one variable, as noted by Fliess. In any case, shuffle product is very useful in combinatorics.

9. Cohn's free field

The ring of rational series (with its noncommutative product) is by definition the ring containing the noncommutative polynomials, and containing the inverse of each series which has nonzero constant term. It is however not a (skew) field; for example, the variables have no inverse in this ring.

There exist however fields which contain noncommutative polynomials. Among these, there is one which is canonical, and has the "less possible relations". It was constructed by P.M. Cohn and called the free field Cohn (1985). It is not easy to describe. One construction uses Malcev–Neumann series on the free group generated by A. These series (in a generalized sense), form a very big field, that contains the non-commutative polynomials. Its subfield generated by the latter is the free field. A natural question arises: indeed, rational series are in the free field; conversely, if a series, in the usual sense, is in the free field, is it a rational series? This was positively answered by Fliess (1970), thereby clarifying the picture.

10. From the Fliess–Hankel matrix to Connes's criterion

The Fliess–Hankel matrix criterion says that a series S is rational if and only if its translates are of finite rank. The translate of S with respect to the word u is the series $u^{-1}S = \sum_w (S, uw)w$. For example, in the one variable case,

$$x^{-1}\left(\sum_{n\geq 0} a_n x^n\right) = \sum_{n\geq 0} a_{n+1} x^n.$$

There have been several unsuccessful attempts to generalize this result to Malcev–Neumann series. Finally, by using a construction of Connes, I could with Duchamp generalize the criterion of Fliess to these series.

The construction of Connes goes as follows, Connes (1994). Let Γ be the free group generated by A and Γ^1 denote the set of edges of its Cayley graph with set of generators A: an edge is therefore a 2-element set $\{g, h\}$ of elements of Γ such that $h = ga$, $a \in A \cup A^{-1}$. Let E (resp. $\bar{E}$) denote the K-vector space of formal linear combinations (resp. of formal infinite linear combinations) of elements of Γ and Γ^1. Connes defines an operator F on E and $\bar{E}$, which sends each elements $h \in \Gamma \backslash \{1\}$ onto the edge $\{g, h\}$, where the length of g is one less than that of h; conversely, such an edge is mapped onto h; the neutral element 1 of Γ is mapped onto zero; F is then naturally extended to (infinite) linear combinations. Moreover, Γ acts naturally on the left on the Cayley graph, hence each series S on the free group defines a mapping $S : E \to \bar{E}$ by left multiplication. Connes then considers the mapping $[F, S] = FS - SF$ from E into $\bar{E}$.

We shall not go into many details, but in essence, Connes proves that if S is rational, then the operator $[F, S]$ has finite rank. He conjectures that the converse is also true. In Duchamp and Reutenauer (1997), I could prove this conjecture. Actually, the calculations show that Connes's condition implies the finiteness of the rank of the translates of S. I thought, when we did this work mid 90s, that I was lucky that the interest of Michel had changed, since this work was exactly the kind of things he was doing 20 years before.

References

J. Berstel, *Transductions and Context-free Languages*, Wiesbaden, Germany: Teubner, 1978.

J. Berstel and C. Reutenauer, *Rational Series and their Languages*, Berlin: Springer, 1988.

K.T. Chen, "Integration of paths, geometric invariants and a generalized Baker-Hansdorff formula", *Ann. Mathe.*, 65, pp. 163–178, 1957.

A. Connes, *Noncommutative Geometry*, New York: Academic Press, 1994.

P.M. Cohn, *Free Rings and their Relations*, 2nd ed., New York: Academic Press, 1985.

P.M. Cohn and C. Reutenauer, "On the construction of the free field", *Inter. J. Algebra and Comput.*, 9, pp. 307–323, 1999.

G. Duchamp and C. Reutenauer, "Un critère de rationalité provenant de la géométrie noncommutative", *Inventiones Mathe.*, 128, pp. 613–622, 1997.

S. Eilenberg, *Automata, Languages and Machines*, Vol. A, New York: Academic Press, 1974.

M. Fliess and J. Richard, "Unités d'algèbres de Hurwitz de séries à une indéterminée", *Comptes Rendus de l'Académie des Sciences*, Paris, Série A–B, 268, pp. 1258–1260, 1969.

M. Fliess, "Du produit de Hurwitz de deux séries formelles", *Comptes Rendus de l'Académie des Sciences*, Paris, Série A–B, 268, pp. 535–537, 1969.

M. Fliess, "Sur le plongement des séries rationnelles non commutatives dans un corps gauche", *Comptes Rendus de l'Académie des Sciences*, Paris, Série A–B, 271, pp. 926–927, 1970.

M. Fliess, "Sur la définition des séries algébriques", *Comptes Rendus de l'Académie des Sciences*, Paris, Série A–B, 273, 284–287, 1971.

M. Fliess, "Sur la réalissation des systèmes dynamique bilinéaires", *Comptes Rendus de l'Académie des Sciences*, Paris, Série A, 277, pp. 923–926, 1973.

M. Fliess, "Matrices de Hankel", *J. de Mathé. Pures et Appliquées*, 53, pp. 197–222, 1974a.

M. Fliess, "Sur divers produits de séries formelles", *Bulletin de la Société mathématique de France*, 102, pp. 181–191, 1974b.
M. Fliess, "Séries rationnelles positives et processus stochastiques", *Annales de l'Institut Henri Poincaré*, Section B, 11, pp. 1–21, 1975.
R. Ree, "Lie elements and an algebra associated with shuffles", *Ann. Mathe.*, 68, pp. 210–220, 1958.
C. Reutenauer, *Free Lie Algebras*, Oxford: Oxford University Press, 1993.
C. Reutenauer, "Inversion height in free fields", *Selecta mathematica*, 2, pp. 93–109, 1996.
M. Sweedler, *Hopf Algebras*, Menlo Park, USA: Benjamin, 1969.
A. Salomaa and M. Soittola, *Automata-Theoretic Aspects of Formal Power Series*, Berlin: Springer, 1978.