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Generalized descent patterns in permutations and associated Hopf algebras

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ABSTRACT

Descents in permutations or words are defined from the relative position of two consecutive letters. We investigate a statistic involving patterns of k consecutive letters, and show that it leads to Hopf algebras generalizing noncommutative symmetric functions and quasi-symmetric functions.

1. Introduction and Background

There are many examples of graded Hopf algebras based on combinatorial structures. These structures can generally be encoded by special classes of words, *e.g.*, permutations [10,2], parking functions [12], Young tableaux [15,2], ordered set partitions [13,14], or binary trees [9,5]. These algebras are often informally called *combinatorial Hopf algebras*. The simplest examples are the mutually dual algebras **Sym** (Noncommutative Symmetric Functions [3]) and *QSym* (Quasi-symmetric functions [4]), based on integer compositions, which can be interpreted as recording descent patterns of permutations. The aim of this paper in to generalize the notion of descent pattern and the associated Hopf algebras.

Recall that the *standardized word* $\mathrm{std}(w)$ of a word $w \in A^*$ over a totally ordered alphabet A is the permutation obtained by iteratively scanning w from left to right, and labeling $1, 2, \ldots,$ the occurrences of its smallest letter, then numbering the occurrences of the next smallest one, and so on. For example, $\mathrm{std}(bbacab) = 341{,}625$.

A permutation $\sigma \in \mathfrak{S}_n$ is said to have a *descent* at i if $\sigma(i) > \sigma(i+1)$. One can alternatively say that the standardization of the two-letter word $\sigma(i)\sigma(i+1)$ is 21, and the descent set of σ can be encoded by the *descent pattern* $(\mathrm{std}(\sigma(i)\sigma(i+1)))_{i=1\ldots n-1}$, produced by scanning σ with a sliding window

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of width two. For example, the permutation $\sigma = 85{,}736{,}124$ has as descent pattern the sequence $(21, 12, 21, 12, 21, 12, 12)$.

A natural generalization of this notion is to use a window of arbitrary width k. Define the *k-descent pattern* of σ as $p^{(k)}(\sigma) = (p_1, \dots, p_{n-k+1})$, where $p_i = \mathrm{std}(\sigma(i)\sigma(i+1)\dots\sigma(i+k-1))$, for $i = 1, \dots, n-k+1$. For example, the permutation $\sigma = 85{,}736{,}124$ has as 3-descent pattern the sequence

$$p^{(3)}(\sigma) = (312, 231, 312, 231, 312, 123). \quad (1)$$

This idea immediately raises some questions. It is known that the sums of permutations of $\mathfrak{S}_n$ having the same descent pattern span a subalgebra Σ_n of the group algebra (Solomon's descent algebra [18]) and that the direct sum of all the Σ_n has a natural Hopf algebra structure (noncommutative symmetric functions or equivalently, the descent algebra), inherited from that of the Hopf algebra of permutations [10,7,2] and of the free algebra. Are there analogs of these facts for the generalized descent classes?

We will show that, although the k-descent classes do not span a subalgebra of the group algebra, the Hopf algebra construction still works, and leads to some interesting combinatorics.

2. Generalized descent patterns and codes

For ordinary descents in $\mathfrak{S}_n$, a classical encoding is by compositions of n. Recall that if the descents of σ form the set $D = \{d_1, \dots, d_{r-1}\}$, we encode it by the composition $I = (i_1, \dots, i_r)$ of n such that $d_j = i_1 + i_2 + \dots + i_j$. Note that in the algebra of noncommutative symmetric functions, the concatenation of compositions corresponds to the multiplication in various bases, appropriately called *multiplicative bases*, since the product of two such elements gives rise to only one element.

2.1. The k-descent code

The most classical encoding of the descents of a permutation $\sigma \in \mathfrak{S}_n$ is a sequence of $+$ and $-$ of length $n-1$. It is equal to $b = b_1 b_2 \cdots b_{n-1}$ where $b_i = -$ if $\sigma(i) > \sigma(i+1)$ and $b_i = +$ otherwise. For example, if $\sigma = 85{,}736{,}124$, $b = -+-+-++$. We generalize this construction (actually a slight variant of it) to k-descent patterns. We regard $\sigma \in \mathfrak{S}_n$ as a permutation of Z which is the identity outside $[1, n]$. We associate with it the word $d_1 \cdots d_n$ over the alphabet $[1, n]$, where d_i is the relative position of $\sigma(i)$ w.r.t. its $k-1$ predecessors $\sigma(i-1), \dots, \sigma(i-k+1)$: that is, $d_i = j$ if $\sigma(i)$ is the jth element of the set $\{\sigma(i), \sigma(i-1), \dots, \sigma(i-k)\}$, naturally ordered. We now define

$$\mathrm{DC}_k(\sigma) = d_1 d_2 \cdots d_n \in [1, k]^n \quad (2)$$

which we will call the *k-descent code*.

We see that, for $k = 2$, we have $d_i = 1$ if $\sigma(i-1) > \sigma(i)$ and $d_i = 2$ otherwise. For example, we obtain $\mathrm{DC}_2(85{,}736{,}124) = 21{,}212{,}122$ and we recover the sequence b above by removing d_1 (always equal to 2) and identifying 1 with $-$, and 2 with $+$. For $k = 3$, we have $\mathrm{DC}_3(85{,}736{,}124) = 32{,}212{,}123$. Both operations $p^{(k)}$ and DC_k do not look the same, but actually:

Proposition 2.1. *Let σ be a permutation. The k-descent pattern $p^{(k)}(\sigma)$ of σ is equivalent to the k-descent code $\mathrm{DC}_k(\sigma)$ of σ: one can deduce one from the other.*

Proof. Starting from the k-descent pattern $p = (p_1, \dots, p_{n-k+1})$ of σ, one recovers the k-descent code by first computing the k-descent code of p_1 and appending to it the final letters of all words p_2 up to p_{n-k+1}; in other words, $\mathrm{DC}_k(\sigma) = \mathrm{DC}_k(p_1)\mathit{last}(p_2) \cdots \mathit{last}(p_{n-k+1})$, where $\mathit{last}(u)$ is the last letter of the nonempty word u.

Conversely, $\mathrm{DC}_k(\sigma) = d_1 \dots d_n$ gives back $p^{(k)}(\sigma)$; indeed, the sequence $d_1 \dots d_k$ is equal to the complement to k of the Lehmer code $L(p_1)$ of p_1 (obtained by changing each i into $k-i$ in the word $L(p_1)$), and then, for $j = 2, \dots, n-k+1$, it gives back p_j from p_{j-1} and d_{j+k-1}: indeed, let u be the word obtained by removing the first letter of p_{j-1}; define v by $\mathrm{std}(v) = \mathrm{std}(u)$ with the alphabet of

v being equal to $\{1, \ldots, k\} \setminus d_{j+k-1}$ and finally $p_j = v.d_{j-k+1}$. Recall that the *Lehmer code* of $\alpha \in \mathfrak{S}_k$ is $L(\alpha) = a_1 \cdots a_k$ with $a_j = |\ \{i \mid i < j, \alpha(i) > \alpha(j)\}\ |$. Details of the verification are left to the reader. □

For example, one can check on $\mathrm{DC}_3(85{,}736{,}124) = 32{,}212{,}123$ that the above algorithm gives back sequence (1).

Here are two more examples of the k-descent code.

$$\mathrm{DC}_3(426135) = 323123 \quad \text{and} \quad \mathrm{DC}_4(426135) = 434133. \tag{3}$$

Note that the definition of $p^{(k)}$ does not require σ to be a permutation and that, on any word over an ordered alphabet A, we have $p^{(k)}(w) = p^{(k)}(\mathrm{std}(w))$. We also extend the k-descent code in a straightforward way to words by setting $\mathrm{DC}_k(w) = \mathrm{DC}_k(\mathrm{std}(w))$.

2.2. The k-recoil code and equivalence classes

The k-descent code of the inverse permutation will be called the *k-recoil code*: $\mathrm{RC}_k(\sigma) := \mathrm{DC}_k(\sigma^{-1})$. It can be computed without inverting permutations in the following way: first see $\sigma \in \mathfrak{S}_n$ as a permutation of Z as before. Then, for all i, restrict σ to $[i-k+1, i]$ (we call *restriction* to a subalphabet $B \subset A$ of a word w on A the image of w under the homomorphism which erases the letters not in B, and keeps those in B). Denote by b_i the position of i in this new word. Then $\mathrm{RC}_k(\sigma) = b_1 \cdots b_n$. For example, $\mathrm{RC}_3(425{,}163) = 323{,}123$; indeed, for example, the 5-th letter is 2 since the restriction of 425163 to $\{3, 4, 5\}$ is 453. Note that this example is coherent with the previous example on the k-descent code since $425{,}163^{-1} = 426{,}135$.

Since we only need to compare letter i with the $k - 1$ preceding letters, we can rephrase this construction with the help of the standardization process. Indeed, two permutations σ and τ in $\mathfrak{S}_n$ have the same k-recoil code if and only if

$$\forall i \le n - k + 1, \quad \mathrm{std}(\sigma_{|_{[i,i+k-1]}}) = \mathrm{std}(\tau_{|_{[i,i+k-1]}}), \tag{4}$$

where $\sigma_{|_{[a,b]}}$ denotes the restriction of σ to the interval $[a, b]$.

Note that, again, the definition of the k-recoil code can be extended to words by setting $\mathrm{RC}_k(w) = \mathrm{RC}_k(\mathrm{std}(w))$. Let us define an equivalence relation on words as $u \equiv_k v$ if, and only if u is a rearrangement of v and u and v have same k-recoil code. In particular, in $\mathfrak{S}_n$, two permutations are equivalent if, and only if they have same k-recoil code. For $k = 2$, this relation is known as the hypoplactic relation (see [8,11]). We shall see that this relation for general k is compatible with the concatenation, hence providing us with a structure of a monoid.

For example, with $k = 3$, each equivalence class in $\mathfrak{S}_n$ with $n \le 3$ has only one element. In $\mathfrak{S}_4$, there are 18 classes, among which are 6 non-singleton classes:

$$[1423, 4123],\ [1432, 4132],\ [2143, 2413],\ [2314, 2341],\ [3142, 3412],\ [3214, 3241]. \tag{5}$$

For example, the 3-recoil code of both 2314 and 2341 is 3223.

Note that the first letter of the k-descent (or recoil) code is always k and the next one is either k or $k - 1$. More generally, a k-recoil code is a word $I = (i_1, \ldots, i_n)$ satisfying $i_\ell \in [\max(k - \ell + 1, 1), k]$ for all ℓ. Conversely, given a word I satisfying these conditions, one easily builds a permutation with I as k-recoil code. By induction, there exists a permutation σ with k-recoil code $I' = (i_1, \ldots, i_{n-1})$. Now, place n anywhere between the $(i_n - 1)$-th and the i_n-th of those elements of σ which lie in the interval $[n - k + 1, n - 1]$. Then one obtains a permutation which has I as k-recoil code. Note that in particular this allows one to build easily the smallest (resp. the largest) elements for the lexicographic order of each equivalence class: put at each step letter r at the rightmost (resp. leftmost) possible slot.

Proposition 2.2. *The k-descent and k-recoil codes $I = (i_1, \ldots, i_n)$ are characterized by the following inequalities:*

$$\begin{cases} k - \ell + 1 \le i_\ell \le k & \text{if } \ell \le k, \\ 1 \le i_\ell \le k & \text{if } \ell \ge k. \end{cases} \tag{6}$$

In particular, the number $\mathrm{N}(k,n)$ *of* k*-descent (or recoil) classes of* $\mathfrak{S}_n$ *is equal to*

$$\begin{cases} n! & \text{if } n \le k, \\ k!\,k^{n-k} & \text{if } n \ge k. \end{cases} \qquad \square \tag{7}$$

For example, the 3-recoil codes of all permutations of $\mathfrak{S}_3$ and $\mathfrak{S}_4$ are, taking the permutations in lexicographic order:

$$333, 332, 323, 322, 331, 321. \tag{8}$$

$$3333, 3332, 3323, 3322, 3331, 3321, 3233, 3232, 3223, 3223, 3232, 3222,$$
$$3313, 3312, 3213, 3213, 3312, 3212, 3331, 3321, 3231, 3221, 3311, 3211. \tag{9}$$

In particular, one can check that the codes 3331, 3321, 3232, 3223, 3312 and 3213 occur twice, and correspond to the six non-singleton 3-classes in $\mathfrak{S}_4$ (see Eq. (5)).

2.3. Classes of permutations having the same k-recoil code

The following proposition generalizes the well-known fact that two permutations have the same recoils (descents of their inverses) if and only if one can go from one to the other by successively exchanging adjacent values whose difference is at least 2 in absolute value.

Proposition 2.3. *Two permutations* σ *and* μ *have the same* k*-recoil code if and only if one can go from* σ *to* τ *by successively exchanging adjacent values whose difference is at least* k*.*

Proof. Let us write $\sigma \equiv'_k \tau$ if one can go from σ to τ by exchanging adjacent values whose difference is at least k. Then it follows from Eq. (4) that

$$\sigma \equiv'_k \tau \Rightarrow \sigma \equiv_k \tau. \tag{10}$$

So each $\equiv'_k$ class is contained in an $\equiv_k$ class.

Now, each $\equiv'_k$ class has a minimal element for the lexicographic order; such a permutation σ has the following property: for each i, $\sigma(i) < \sigma(i+1)+k$. Let us denote by $W(k,n)$ the set of permutations in $\mathfrak{S}_n$ having this property.

Observe there are at most as many $\equiv'_k$-classes as elements in $W(k,n)$. Moreover, if one removes n from any word of $W(k,n)$, one obtains a word in $W(k,n-1)$. Conversely, given a word of $W(k,n-1)$, in order to get a word in $W(k,n)$, one can put n at n different spots if $n \le k$ or at k different spots (before $n-k+1, \ldots, n-1$ or at the end) if $n \ge k$. Hence $|W(k,n)|$ is equal to $\mathrm{N}(k,n)$, the number of $\equiv_k$ classes thanks to the formula relating $\mathrm{N}(k,n)$ and $\mathrm{N}(k,n-1)$.

We deduce that both relations have the same number of classes, hence they coincide. $\square$

This argument unravels a simple characterization of the minimum element of an equivalence class.

Corollary 2.4. *The minimum elements of the classes of* $\equiv_k$ *are the permutations* σ *such that for any i, one has* $\sigma(i) - \sigma(i+1) < k$*.*

Symmetrically, the maximum elements of $\equiv_k$ *are the permutations* σ *such that for any i, one has* $\sigma(i+1) - \sigma(i) < k$*.*

Moreover, the set of maximal elements of $\mathfrak{S}_n$ *is obtained from the set of minimal elements of* $\mathfrak{S}_n$ *by the transformation replacing in a permutation each value i by* $n+1-i$*.* $\square$

Recall that the *right permutohedron* is the set $\mathfrak{S}_n$ with the *right weak (Bruhat) order*; the latter is the reflexive and transitive closure of the relation defined by $\sigma < \alpha$ if $\alpha = \sigma\tau$ for some adjacent transposition τ with $l(\sigma) < l(\alpha)$. The *left weak order* is defined symmetrically. For later use, call *inversion set* of a permutation σ the set $I(\sigma)$ of all pairs $(\sigma(i), \sigma(j))$ such that $i < j$ and $\sigma(i) > \sigma(j)$. It is well-known that the mapping $\sigma \mapsto I(\sigma)$ is an order isomorphism between the right permutohedron and the set of inversion sets of permutations ordered by inclusion.

Given a k-recoil class C, denote by α_C (respectively ω_C) its smallest (resp. greatest) element in lexicographic order. The previous proposition implies that the $\equiv_k$ classes split the right permutohedron into connected components of its Hasse diagram. We can be more precise.

Proposition 2.5. *The set of permutations having a given k-recoil (respectively descent) code is an interval of the right (resp. left) weak order.*

Proof. Let C be a k-recoil class. By Corollary 2.4, we have $C \subset [\alpha_C, \omega_C]$.

For the converse, let $\sigma \in [\alpha_C, \omega_C]$. Then its inversion set is contained in the inversion set of ω_C and contains the inversion set of α_C. This shows that for any i, the letters $i, i+1, \ldots, i+k-1$ in σ and α_C have the same relative positions, since this is true for α_C and ω_C. Hence σ has the same k-recoil code as α_C and lies therefore in C. $\square$

As in the case of ordinary descents, the order ideals defined by maximal elements are unions of classes. This property is essential for defining multiplicative bases in the associated Hopf algebras defined later.

Proposition 2.6. *The interval* $[\mathrm{id}, \omega_C]$ *of the right permutohedron is a union of* $\equiv_k$ *classes, where* id *is the identity permutation.*

Moreover, the interval $[\alpha_C, \omega]$ *of the right permutohedron is also a union of* $\equiv_k$ *classes, where* ω *is the longest permutation in* $\mathfrak{S}_n$.

Proof. By Corollary 2.4, the second statement is equivalent to the first one.

Thus, we must prove that if $x \le \omega_C$ then the maximal element $\omega_{C'}$ of the $\equiv_k$ class C' of x satisfies $\omega_{C'} \le \omega_C$. If x is maximal in its $\equiv_k$-class, then $x = \omega_{C'}$ and we are done. Otherwise, thanks to the characterization of the maximal elements, we have

$$x = \ldots ij \ldots, \tag{11}$$

where $j - i \ge k$. We claim that the permutation x' obtained from x by exchanging i and j also satisfies $x' \le \omega_C$. Then x' is in C' and we see by induction on the distance from x to $\omega_{C'}$ (which is the difference of their lengths) that $\omega_{C'}$ satisfies $\omega_{C'} \le \omega_C$. For the claim, consider the subset of the elements of the right permutohedron greater than x such that (j, i) is not an inversion; that is, by the characterization of the right order through the inversion set, the set of elements greater than or equal to x and not greater than x'. This set does not contain any maximal element of a $\equiv_k$-class (and hence this set does not contain $\omega_{C'}$, which implies that $x' \le \omega_{C'}$). Indeed, the values between i and j in such permutations can only be either smaller than i or greater than j: this is true for x, and is proved inductively. Hence there are always two consecutive values a, b with difference $b - a$ at least k. $\square$

The proof of the previous proposition relies on the fact that if $x < y$ then the maximal element of the class of x is smaller than or equal to the maximal element of the class of y. Let us summarize:

- the weak order is a lattice,
- k-recoil classes are intervals of the weak order,
- the two following relations on k-recoil classes are both orders and the same: $C < C'$ if an element of C is smaller (weak order) than an element of C'; $C <' C'$ if $\omega_C < \omega_{C'}$.

Put together, these three facts are the assumptions of a theorem by Reading [16] whose claim is that the order $C < C'$ defined previously is a lattice. Hence

Corollary 2.7. *The order $<$ defined among $\equiv_k$ classes by $C < C'$ if $\omega_C < \omega_{C'}$ is a lattice.*

2.4. A monoid structure on words

Recall that two words are $\equiv_k$ equivalent if, and only if their standardized words are, and that two permutations are equivalent if, and only if one can go from one to the other by successively exchanging adjacent values whose difference is at least k. We then have:

$$u \equiv_k u' \quad \text{and} \quad v \equiv_k v' \Longrightarrow u \cdot v \equiv_k u' \cdot v', \tag{12}$$

where $\cdot$ denotes the concatenation of words. Indeed, consider $\sigma = \mathrm{std}(u \cdot v)$ and $\sigma' = \mathrm{std}(u' \cdot v')$. Then the prefix of σ of size $|u|$ is a rearrangement of the prefix of τ of size $|u|$ since $\equiv_k$ works inside rearrangement classes. Moreover, one can mimic the exchanges of values going from $\mathrm{std}(u)$ to $\mathrm{std}(u')$ to go from $\mathrm{std}(u \cdot v)$ to $\mathrm{std}(u' \cdot v)$ since the differences between those values are greater than or equal to the differences on $\mathrm{std}(u)$. Hence $u \cdot v \equiv_k u' \cdot v$. The other part is done in the same way.

So the quotient of the free monoid by $\equiv_k$ defines a monoid structure analogous to the hypoplactic monoid. Since there is a monoid for each k, we get an interpolation between the hypoplactic monoid ($k = 2$) and the free monoid ($k = \infty$). For all of these, it is possible to define a Robinson–Schensted-like correspondence. Finally, note that these congruences are compatible with restriction to alphabet *intervals*, that is: if B is an interval of A and $u \equiv_k v$ then $u_{|_B} \equiv_k v_{|_B}$.

2.5. k-Eulerian polynomials and k-major index

The classical Eulerian polynomials count permutations by their number of descents, or equivalently, by the number of 1's in their 2-descent code.

In this form, the definition can be easily generalized. We define the k-Eulerian polynomial $E_{n,k}$ as the sum over $\mathfrak{S}_n$ of the product of t_j where j runs through all entries of their k-recoil code, except the first one, and except the j with $j = k$. Formally

$$E_{n,k}(t_1, \dots, t_{k-1}) = \sum_{\sigma \in \mathfrak{S}_n} t_{RC_k}(\sigma)$$

where $t_C = \prod_{2 \le i \le n, C_j \ne k} t_{C_j}$, with $C = (C_1, \dots, C_n)$. Note that for $k = 2$, $E_{n,2}(t_1)$ is the Eulerian polynomial.

For example, with $k = 3$, we obtain from (8) and (9)

$$E_{1,3} = 1; \qquad E_{2,3} = t_2 + 1; \qquad E_{3,3} = 1 + 2t_2 + t_2^2 + t_1 + t_1 t_2; \tag{13}$$

$$E_{4,3} = 1 + t_1^2 t_2 + t_1^2 t_3 + 2\, t_1 t_2^2 + 7\, t_1 t_2 + 3\, t_1 + t_2^3 + 5\, t_2^2 + 3\, t_2. \tag{14}$$

The classical major index of a permutation σ is the sum of the positions of the descents of σ. We replace the monomial $q^{\mathrm{maj}(\sigma)}$ by the product $q_C = \prod_{2 \le i \le n, C_i \ne k} q_{C_i}^{i-1}$ where $C = (C_1, \dots, C_n) = \mathrm{RC}_k(\sigma)$. Formally,

$$M_{n,k}(q_1, \dots, q_{k-1}) = \sum_{\sigma \in \mathfrak{S}_n} q_{\mathrm{RC}_k(\sigma)}.$$

Note that for $k = 2$, $M_{n,2}(q_1)$ is the classical enumerator of $\mathfrak{S}_n$ by major index. For example, with $k = 3$, the k-major index polynomials of the first symmetric groups are

$$M_{1,3} = 1; \qquad M_{2,3} = 1 + q_2; \qquad M_{3,3} = 1 + q_1^2 q_2 + q_1^2 + q_2^2 + q_2^3 + q_2. \tag{15}$$

3. Associated combinatorial Hopf algebras

The equivalence relation $\equiv_k$ can be used to define subalgebras and quotients of the Hopf algebra of permutations, generalizing respectively noncommutative symmetric functions [3], or equivalently the descent Hopf algebra [10], and quasi-symmetric functions [4].

3.1. Background on **FQSym**

Let A be a totally ordered infinite alphabet. Recall that the Hopf algebra of permutations, introduced in [10], can be realized as the algebra **FQSym** (free quasi-symmetric functions, cf. [2]), spanned over Z by the polynomials

$$\mathbf{G}_\sigma(A) = \sum_{\mathrm{std}(w) = \sigma} w. \tag{16}$$

Recall from [10,2] that, for $\alpha \in \mathfrak{S}_n$, $\beta \in \mathfrak{S}_p$, the product $\mathbf{G}_\alpha(A)\mathbf{G}_\beta(A)$ is equal to the sum of all $\mathbf{G}_\sigma(A)$ with $\sigma = u \cdot v \in \mathfrak{S}_{n+p}$ (the dot denotes the concatenation of the words u and v) such that $\mathrm{std}(u) = \alpha$ and $\mathrm{std}(v) = \beta$. Moreover the coproduct $\Delta(\mathbf{G}_\sigma)$, $\sigma \in \mathfrak{S}_n$, is equal to the sum of all $\mathbf{G}_\alpha \otimes \mathbf{G}_\beta$, for $i = 0, \dots, n$, where $\alpha = \mathrm{std}(\sigma_{|[1,i]}) = \sigma_{|[1,i]}$ and $\beta = \mathrm{std}(\sigma_{|[i+1,n]})$. As an example, we have

$$\mathbf{G}_{12}\mathbf{G}_{21} = \mathbf{G}_{1243} + \mathbf{G}_{1342} + \mathbf{G}_{1432} + \mathbf{G}_{2341} + \mathbf{G}_{2431} + \mathbf{G}_{3421} \tag{17}$$

and

$$\Delta(\mathbf{G}_{3124}) = \mathbf{G}_{\epsilon} \otimes \mathbf{G}_{3124} + \mathbf{G}_1 \otimes \mathbf{G}_{213} + \mathbf{G}_{12} \otimes \mathbf{G}_{12} + G_{312} \otimes \mathbf{G}_1 + \mathbf{G}_{3124} \otimes \mathbf{G}_{\epsilon}, \tag{18}$$

where ϵ denotes the empty permutation.

Recall from [1] page 9, that **FQSym** has another basis indexed by permutations, denoted by $\mathbf{S}^{\sigma}$ and defined by $\mathbf{S}^{\sigma} = \sum_{\tau \le \sigma} \mathbf{G}_{\tau}$, where $\le$ is the left weak order. Likewise, it has the basis $\mathbf{E}^{\sigma} = \sum_{\tau \ge \sigma} \mathbf{G}_{\tau}$, see [1] page 12, where this basis is denoted $\boldsymbol{\Lambda}^{\sigma}$. Both bases are *multiplicative* (the product of two elements gives rise to only one element). More precisely

$$\mathbf{S}^{\sigma}\mathbf{S}^{\tau} = \mathbf{S}^{\sigma[|\tau|]\cdot\tau}, \qquad \mathbf{E}^{\sigma}\mathbf{E}^{\tau} = \mathbf{E}^{\sigma\cdot\tau[|\sigma|]}, \tag{19}$$

where $\mu[i] = (\mu_1 + i, \ldots, \mu_n + i)$.

3.2. Hopf subalgebras

Imitating the case $k = 2$, we define *generalized ribbons* by

$$\mathsf{R}_C = \sum_{\mathrm{DC}_k(\sigma)=C} \mathbf{G}_{\sigma} \tag{20}$$

for a k-descent code C. We denote by $\mathbf{DSym}^{(k)}$ the submodule of **FQSym** spanned by the elements R_C, for all k-recoil codes C.

Since the k-descent code is the same for a word and its standardized word, we have a simple realization of $\mathbf{DSym}^{(k)}$ as series in noncommuting variables:

$$\mathsf{R}_C = \sum_{\mathrm{DC}_k(w)=C} w. \tag{21}$$

This basis generalizes the classical strict descent classes (sum of permutations whose descent set is equal to a prescribed one).

Now, there is a standard way of proving that certain sums of $\mathbf{G}_{\sigma}$ span a subalgebra of **FQSym**: it consists in looking at these sums as obtained from a monoid with some additional structure. It has to be done on the dual basis $\mathbf{F}_{\sigma}$ of $\mathbf{G}_{\sigma}$, which can be identified with $\mathbf{G}_{\sigma^{-1}}$ (**FQSym** and its graded dual are isomorphic). The general theorem, proven by Hivert and Nzeutchap [6] states

Theorem 3.1 (*[6]*). *Let $\equiv$ be a congruence that is compatible with the restriction to alphabet intervals and with the destandardization process, then the sums*

$$P_{\tau} := \sum_{\sigma;\sigma\equiv\tau} \mathbf{F}_{\sigma} \tag{22}$$

span a Hopf subalgebra of **FQSym**.

We have seen before that $\equiv_k$ is a congruence and that it is compatible with the restriction to alphabet intervals. The compatibility with the destandardization process just states that two words are equivalent if, and only if their standardized words are equivalent and one is a rearrangement of the other. It is true almost by definition since w and $\mathrm{std}(w)$ have the same k-recoil code. Moreover, R_C is of the previous type:

$$\mathsf{R}_C = \sum_{\mathrm{DC}_k(\sigma)=C} \mathbf{G}_{\sigma} = \sum_{\mathrm{RC}_k(\sigma)=C} \mathbf{F}_{\sigma} = \sum_{\sigma\,\equiv_k\,\tau} \mathbf{F}_{\sigma}, \tag{23}$$

for any τ such that $\mathrm{RC}_k(\tau) = C$. Hence

Theorem 3.2. $\mathbf{DSym}^{(k)}$ *is a Hopf subalgebra of* **FQSym**. □

For example, we have

$$\mathsf{R}_{321}\mathsf{R}_{3321} = \mathsf{R}_{3211221} + \mathsf{R}_{3211321} + \mathsf{R}_{3212321} + \mathsf{R}_{3213321}. \tag{24}$$

(the verification of this calculation is left to the reader: write everything in the $\mathbf{G}_\sigma$ basis and use the fact that the products are multiplicity-free). This example shows that, unlike ribbon skew-Schur functions, or noncommutative ribbons functions, the product of two R_C may be equal to a sum of more than two such functions.

Clearly, for $k < l$, $\sigma \equiv_l \tau$ implies $\sigma \equiv_k \tau$, hence.

Corollary 3.3. *$\mathbf{DSym}^{(k)}$ is a Hopf subalgebra of $\mathbf{DSym}^{(l)}$ for $k < l$. In particular, the $\mathbf{DSym}^{(k)}$ interpolate between $\mathbf{Sym} = \mathbf{DSym}^{(2)}$ and $\mathbf{FQSym} = \mathbf{DSym}^{\infty}$.* □

3.3. Duality

Recall that **FQSym** has a scalar product $\langle\, ,\rangle$ for which it is a self-dual Hopf algebra. This scalar product is defined by $\langle \mathbf{G}_\sigma, \mathbf{F}_\tau\rangle = 1$ if $\sigma = \tau$ and $= 0$ otherwise, see [7,2] p. 680. In other words, the **G** and **F** bases are orthogonal bases. It follows from general theorems on bialgebras that the orthogonal of any subalgebra is a bi-ideal. In our case, taking $\mathbf{DQSym}^{(k)}$ as subalgebra, its orthogonal is spanned by the elements $\mathbf{F}_\sigma - \mathbf{F}_\tau$, where σ and τ have the same k-descent set; that is, $\sigma^{-1} \equiv_k \tau^{-1}$.

Let us denote by $\mathbf{DQSym}^{(k)} = \mathbf{DSym}^{(k)*}$, the graded dual bialgebra of $\mathbf{DSym}^{(k)}$. We therefore obtain.

Theorem 3.4. *$\mathbf{DQSym}^{(k)}$ is isomorphic with $\mathbf{FQSym}/\equiv_k$. It is a noncommutative (for $k > 2$) and non-cocommutative Hopf algebra.*

We shall write $\mathsf{F}_C = \overline{\mathbf{F}_\sigma}$, the image of $\mathbf{F}_\sigma$ in $\mathbf{FQSym}/\equiv_k$, where C is the k-descent code of σ. By Prop. 3.4 in [2], the coproduct Δ is given by $\Delta(\mathbf{F}_\sigma) = \sum_{\sigma=u.v} \mathbf{F}_{\mathrm{std}(u)} \otimes \mathbf{F}_{\mathrm{std}(v)}$. As an example, $52413 \equiv_3 21543$, since using Proposition 2.3: $52413 \equiv_3 25413 \equiv_3 25143 \equiv_3 21543$. Hence their inverses 42531 and 21543 have same 3-descent code. We have

$$\begin{aligned}\Delta\mathbf{F}_{42531} &= \mathbf{F}_{42531} \otimes 1 + \mathbf{F}_{3142} \otimes \mathbf{F}_1 + \mathbf{F}_{213} \otimes \mathbf{F}_{21} \\ &\quad + \mathbf{F}_{21} \otimes \mathbf{F}_{321} + \mathbf{F}_1 \otimes \mathbf{F}_{2431} + 1 \otimes \mathbf{F}_{42531},\end{aligned} \tag{25}$$

and

$$\begin{aligned}\Delta\mathbf{F}_{21543} &= \mathbf{F}_{21543} \otimes 1 + \mathbf{F}_{2143} \otimes \mathbf{F}_1 + \mathbf{F}_{213} \otimes \mathbf{F}_{21} \\ &\quad + \mathbf{F}_{21} \otimes \mathbf{F}_{321} + \mathbf{F}_1 \otimes \mathbf{F}_{1432} + 1 \otimes \mathbf{F}_{21543}.\end{aligned} \tag{26}$$

Moreover, since the product of the functions $\mathbf{F}_\sigma$ is given by the shifted shuffle product (see [2] Prop. 3.2), we have

$$\mathbf{F}_{42531}\mathbf{F}_1 = \mathbf{F}_{425316} + \mathbf{F}_{425361} + \mathbf{F}_{425631} + \mathbf{F}_{426531} + \mathbf{F}_{462531} + \mathbf{F}_{642531}, \tag{27}$$

and

$$\mathbf{F}_{21543}\mathbf{F}_1 = \mathbf{F}_{215436} + \mathbf{F}_{215463} + \mathbf{F}_{215643} + \mathbf{F}_{216543} + \mathbf{F}_{261543} + \mathbf{F}_{621543}, \tag{28}$$

and one easily checks that the indices of both expressions match in a one-to-one correspondence by the $\equiv_3$ relation on the inverse permutations, or, an easier computation by hand, by the $p^{(3)}$ algorithm on permutations themselves.

3.4. Multiplicative bases

As in the case of noncommutative symmetric functions, we can also generalize the weak descent classes (sum of permutations whose descent set is contained in a prescribed one). We set

$$\mathsf{S}^C := \mathbf{S}^{\omega_C^{-1}} \quad \text{and} \quad \mathsf{E}^C := \mathbf{E}^{\alpha_C^{-1}}. \tag{29}$$

Note that the S, the E, and the R span the same submodule of **FQSym** thanks to Proposition 2.6: each S and each E is a sum over an union of DC_k classes, hence a sum of R. Moreover, having defined a partial order that is even a lattice on equivalence classes proves that the transition matrix between the

S and the R is unitriangular if one writes the maximal elements of each class in lexicographic order. The same holds for the E.

Recall that a permutation in $\mathfrak{S}_n$ is *connected* (or *indecomposable*) if it does not belong to any proper Young (or parabolic) subgroup of $\mathfrak{S}_n$. Also, call *mirror image* of a word the word obtained by reading it backwards. We can then state:

Theorem 3.5. *The* S^C*'s and the* E^C*'s form multiplicative bases of* $\mathbf{DSym}^{(k)}$.

Moreover, $\mathbf{DSym}^{(k)}$ *is, as an algebra, freely generated by the* $\mathbf{S}^\sigma$*'s (respectively* $\mathbf{E}^\sigma$*'s) indexed by permutations that are both mirror images of connected permutations (resp. connected permutations) and inverses of maximum (resp. minimum) elements of an* DC_k*-equivalence class.*

Note that the $\mathbf{E}^\sigma$ indexed by connected permutations are natural algebraic generators of **FQSym** [2] and since this basis is multiplicative, their dual elements are a basis of the primitive lie algebra of **FQSym**.

Proof. We already noted that the S^C and E^C form bases of $\mathbf{DSym}^{(k)}$. Since this algebra is a subalgebra of **FQSym** and that these bases are special cases of two multiplicative bases of **FQSym**, the S^C and the E^C are both multiplicative.

Moreover, by [1] the $\mathbf{E}^\sigma$'s, for all permutations σ that are the mirror image of a connected permutation, freely generate **FQSym**. Hence the subalgebra $\mathbf{DSym}^{(k)}$ is free over the set of $\mathbf{E}^\sigma$'s, where σ is both a mirror image of a connected permutation and the inverse of a minimal (for the left weak order) element of a DC_k class (since these minimal elements are precisely the α_C^{-1}s). The same holds for the $\mathbf{S}^\sigma$'s. □

Note that the definition of these particular bases is another way to prove that the algebra $\mathbf{DSym}^{(k)}$ is a Hopf subalgebra of **FQSym**: one first proves that the E^C generate the same vector space as the R^C and then prove that the shifted concatenation of two permutations that are inverses of minimum of classes is also the inverse of a minimum of a class.

3.5. Hilbert series

The Hilbert series of $\mathbf{DSym}^{(k)}$ is, by Proposition 2.2,

$$H_k(t) = \sum_{j=0}^{k-1} j!t^j + \frac{k!t^k}{1-kt}. \tag{30}$$

In order to obtain the generating function $G_k(t)$ of a free homogeneous generating set, we use the equation

$$\frac{1}{1-G_k(t)} = H_k(t). \tag{31}$$

With $k = 3$, one finds

$$\begin{aligned} G_3(t) &= t + t^2 + 3t^3 + 7t^4 + 17t^5 + 41t^6 + 99t^7 + 239t^8 + 577t^9 \\ &\quad + 1393t^{10} + 3363t^{11} + 8119t^{12} + 19601t^{13} + \cdots \end{aligned} \tag{32}$$

whose coefficients form Sequence A001333 of [17]: these coefficients are the numerators of the continued fraction convergents of $\sqrt{2}$. For $k = 4$,

$$\begin{aligned} G_4(t) &= t + t^2 + 3t^3 + 13t^4 + 47t^5 + 173t^6 + 639t^7 + 2357t^8 + 8695t^9 \\ &\quad + 32077t^{10} + 118335t^{11} + 436549t^{12} + 1610471t^{13} + \cdots \end{aligned} \tag{33}$$

whose coefficients form Sequence A084519 of [17].

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