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Inverses of Words and the Parabolic Structure of the Symmetric Group

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We use a bijection from the set of words onto the set of multisets of primitive circular words, to find a construction of the inverse of a word having the properties required by Foata and Han. Moreover, we show the link of this construction with the parabolic structure of the symmetric group, seen as a Coxeter group.

1. INTRODUCTION

Answering a private question of Knuth, Foata and Han [2] give a construction on words, generalizing the inverse of permutations and show that their construction of the inverse of a word has several properties: in particular, it reverses the set of biletters of the word (definition below). They note, however, that their construction does not preserve the number of inversions of the word (as it does for permutations); they leave this as an open question. In a subsequent paper, Clarke [1] gives a construction which has all the required properties, but only for a 3-letter alphabet.

We give here a construction of the inverse of a word in full generality, which has all the properties required by Foata and Han. It rests on the Schensted [6] notion of standardization of a word (which allows him to extend his correspondence from permutations to words). The standard permutation of a word leads to a mapping which associates to a each word w a multiset of circular words (by replacing in the cycles of the permutation each digit by the corresponding letter in w). It is shown in [4] that this mapping is actually bijective; this constitutes a combinatorial version of the theorem of Poincaré–Birkhoff–Witt (actually Lyndon words, or Hall words, would serve as well, but the corresponding bijection has not good functorial properties).

Circular words may be reversed and modulo the previous bijection, this gives our construction of the inverse of a word. See Section 2 and the proof in Section 3.

Besides answering the question of Foata and Han, our construction has interesting links with the parabolic structure of the symmetric group. Indeed, each word w has a unique non-decreasing rearrangement $\overline{w}$, and the stabilizer of the latter, in the usual action of the symmetric group, is a parabolic subgroup or Young subgroup W_I. What makes our construction non-trivial is that the standard permutation of the inverse is not the inverse of the standard permutation. However, we show that these two permutations are conjugated by an element of the parabolic subgroup W_I. Moreover, these two permutations coincide exactly when the word is minimal (a notion introduced by Foata and Han), and in this case, the word is the shortest one in its double coset modulo $W_I \times W_I$.

2. THE INVERSE OF A WORD

Let w be a word of length n on a totally ordered alphabet A and σ a permutation in S_n. We define $w\sigma$ to be the word

$$w\sigma = w_{\sigma(1)} \dots w_{\sigma(n)},$$

where $w = w_1 \dots w_n$. This defines a right action of S_n onto the set A^n of words of length n. For fixed w, a word of the form $w\sigma$ is called a *rearrangement* of w. Among all rearrangements of w, there is one, called its *nondecreasing rearrangement*, which satisfies

$$w_{\sigma(1)} \leq \cdots \leq w_{\sigma(n)}.$$

We denote it by $\overline{w}$. Among all permutations σ such that $w = \overline{w}\sigma$, there is one which is the shortest one (that is, which has the fewest possible inversions), which is called the *standard permutation* of w, and denoted $st(w)$. Standardization was actually introduced by Schensted in [6, p. 189]; it goes as follows: assume that the letters appearing in w are $a, b, \ldots,$ and that $a < b < \cdots$. Then $\sigma = st(w)$, viewed as a word, is obtained as follows: number from 1 to n each letter in w, starting with the a terms, which are numbered from left to right, continuing in the same way with the b terms, and so on.

Inversions of permutations are extended to words as follows: an *inversion* of w is a couple (i, j) such that $i < j$ and $w_i > w_j$. Similarly, a *descent* of w is an index i such that $w_i > w_{i+1}$. It is well known, and easily verified, that inversions (resp. descents) of w and $st(w)$ coincide.

The couple $(\overline{w}, w)$ may canonically be identified with a word of length n on the product alphabet $A \times A$. We shall use this identification and count the number of occurrences of a *biletter* (a, b) in $(\overline{w}, w)$; by abuse, we also call *biletters of* w the biletters of $(\overline{w}, w)$.

Two words u, v are called *conjugate* if for some words x, y, one has $u = xy$, $v = yx$. This defines an equivalence relation on the set of words. A conjugation class is called a *circular word*. We denote by (w) the conjugation class of w. If the cardinality of (w) is equal to the length of w, we say that w and (w) are *primitive*. We call *cycle* a primitive circular word.

We describe now a bijection ϕ from words onto multisets of cycles. Let w be a word and σ its standard permutation. For each cycle $(i_1, \ldots, i_k)$ of σ, the circular word $(w_{i_1} \ldots w_{i_k})$ does not depend on the previous representation of the cycle. By taking all cycles of σ, we obtain a multiset of circular words, denoted $\Phi(w)$. Then Φ is a bijection from words onto multisets of cycles (see [4]).

Given a cycle $c = (a_1 \ldots a_k)$, we denote its *mirror image* by $\mu(c) = (a_k \ldots a_1)$. This mapping μ induces an involution on the set of multisets of cycles.

We denote $w' = \Phi^{-1}\mu\Phi(w)$, called the *inverse* of w.

THEOREM 2.1. *The following properties hold:*

1. *If w is a permutation (viewed as a word), then $w' = w^{-1}$;*
2. *w' is a rearrangement of w;*
3. *$(w')' = w$;*
4. *The number of occurrences of biletter (a, b) in $(\overline{w}, w)$ is equal to the number of occurrences of biletter (b, a) in $(\overline{w}, w')$;*
5. *w and w' have the same number of inversions.*

Let us briefly indicate the construction of Φ^{-1}. Let M be a multiset of cycles. For simplicity, assume that M has no multiplicities, that is, is a set.

Denote by $\widetilde{u}$ the mirror image of u, and by u^∞ the right infinite word obtained by repeating u infinitely often. Let

$$E = \{u^\infty | (\widetilde{u}) \in M\}.$$

The shift mapping $s : a_0a_1a_2 \ldots \to a_1a_2a_3 \ldots$ acts bijectively on E since its elements are periodic. Order the elements of E lexicographically, thus obtaining a unique increasing bijection $\beta : [n] \to E$. Then let $\sigma = \beta^{-1}s^{-1}\beta$.

If $M = \Phi(w)$, then $\sigma = st(w)$, and we recover w by: $w_i = F\beta\sigma(i)$, where Fx is the first letter of the infinite word x.

If M has multiplicities, then one proceeds similarly, ordering first the multiple orbits under the shift, then the elements in these orbits lexicographically. See Section 5 for an example of this construction.

3. PROOF

Note that if w is a word, σ its standard permutation and $c = (i_1, \ldots, i_k)$ a cycle of σ, the associated cycle $(w_{i_1} \ldots w_{i_k})$ equals $(w_{\sigma^{-1}(i_1)} \ldots w_{\sigma^{-1}(i_k)})$, since $c = (\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_k))$.

1. Suppose that w is a permutation. Then $w = st(w) = \sigma$. Thus the cycle associated to c as above is $(i_1 \ldots i_k)$, since $w_i = \sigma(i)$, hence $w_{\sigma^{-1}(i)} = i$. In other words, if w is a permutation, cycles in both meanings coincide, which implies that $w' = w^{-1}$, since inverting a permutation amounts to reverse its cycles.

2. Observe that, for any letter a, the number of occurrences of a in w and in $\Phi(w)$ (in an evident meaning) are equal. Thus the property follows.

3. This is clear since μ is an involution.

Before proving the last two properties, we need a definition: let M be a multiset of cycles; an *m-sequence* of M is a word $a_1 \ldots a_m$ of length m such that for some cycle $c = (u)$ of M, $a_1 \ldots a_m$ is a factor of some power of u. For example, if M contains $c = (aba)$, $baab$ is a 4-sequence of M and aa, ab are 2-sequences. Note that m-sequences in M occur with multiplicities.

4. Let $\sigma = st(w)$; then $\overline{w} = w\sigma^{-1} = w_{\sigma^{-1}(1)} \ldots w_{\sigma^{-1}(n)}$. Hence the multiset of biletters in $(\overline{w}, w)$ is $\{(w_{\sigma^{-1}(i)}, w_i) | i = 1, \ldots, n\}$. On the other hand, by construction of Φ, each 2-sequence of $\Phi(w)$ is of the form $w_{\sigma^{-1}(i)} w_i$, by the remark at the beginning of the section. Thus there is an evident bijection between the multiset of biletters in $(\overline{w}, w)$ and the multiset of 2-sequences in $\Phi(w)$. Since the 2-sequences of $\mu\Phi(w)$ are the reverse of those of $\Phi(w)$, the property follows.

5. We claim that inversions of w are in one-to-one correspondence with couples of m-sequences of $\Phi(w)$, of the form $(a_1 \ldots a_m, b_1 \ldots b_m)$, with $m \geq 2$, $a_1 < b_1$, $a_i = b_i$ for $i = 2, \ldots, m-1$, and $a_m > b_m$.

The claim being admitted, the equality of inversions follows, since the previous property is preserved under reversal.

Let us prove the claim. Note that the m-sequences of $\Phi(w)$ are all of the form $w_{\sigma^{-(m-1)}(i)} \ldots w_{\sigma^{-1}(i)} w_i$, $i = 1, \ldots, n$. Moreover, we have

$$(\star) \qquad i < j \implies (w_{\sigma^{-k}(i)})_{k \geq 1} \leq (w_{\sigma^{-k}(j)})_{k \geq 1}$$

where the sequences in $A^{\mathbb{N}^*}$ are ordered lexicographically. This follows indeed from the construction of the inverse of Φ, see the end of Section 2.

Consider inversion (i, j) of w: we have $i < j$ and $w_i > w_j$. In $(\star)$, we must have strict inequality; otherwise, since the sequences are periodic, we would have $w_i = w_j$. Hence there exists $m \geq 2$ such that $w_{\sigma^{-k}(i)} = w_{\sigma^{-k}(j)}$ for $k = 1, \ldots, m-2$ and $w_{\sigma^{-(m-1)}(i)} < w_{\sigma^{-(m-1)}(j)}$. Thus we find the couple of m-sequences $(w_{\sigma^{-(m-1)}(i)} \ldots w_{\sigma^{-1}(i)} w_i,\ w_{\sigma^{-(m-1)}(j)} \ldots w_{\sigma^{-1}(j)} w_j)$ as in the claim.

Conversely, if we have such a couple, we must have $i \neq j$ (since $w_i > w_j$) and then, $i < j$ (otherwise $i > j$ which by $(\star)$ leads to a contradiction, since $(w_{\sigma^{-k}(i)})_{k \geq 1} < (w_{\sigma^{-k}(j)})_{k \geq 1}$ by assumption on the m-sequences). Thus we obtain inversion (i, j) of w.

4. Parabolic Structure

Recall that a *parabolic subgroup*, or *Young subgroup*, of S_n is a subgroup W_I of S_n generated by a subset I of the set $G = \{(1, 2), (2, 3), \dots, (n-1, n)\}$ of adjacent transpositions, or equivalently, of the form $S_{n_1} \times \cdots S_{n_r}$, $n_1 + \cdots + n_r = n$, canonically embedded in S_n.

We denote by W^I the set of permutations σ whose descent set $D(\sigma)$ is included in $S \setminus I$:

$$W^I = \{\sigma \in S_n \mid \sigma(i) > \sigma(i+1) \implies i \notin I\},$$

once G is naturally identified with $\{1, 2, \dots, n-1\}$.

Equivalently, denote by $\ell(\sigma)$ the *length* of σ, i.e., the number of inversions of σ, or equivalently, the minimum length of a factorization of σ into adjacent transpositions. Then

$$W^I = \{\sigma \in S_n | \ell(\sigma s_i) < \ell(\sigma) \implies i \notin I\},$$

with $s_i = (i, i+1)$. It is a well known fact (see [5]), and valid in each Coxeter group, that W^I is a set of left coset representatives modulo W_I, and more precisely: each σ in S_n has a unique expression as $\sigma = \sigma_0 \pi$ with $\ell(\sigma) = \ell(\sigma_0) + \ell(\pi)$, $\sigma_0 \in W^I$, $\pi \in W_I$ (σ_0 is the unique element of minimal length in σW_I. We also denote ${}^I W = (W^I)^{-1}$. Then, similarly, the set ${}^I W \cap W^I$ is a set of double cosets representatives modulo $W_I \times W_I$.

To each word w, we associate the following parabolic subgroup:

$$W_I = \{\sigma \in S_n \mid \overline{w}\sigma = \overline{w}\},$$

the stabilizer of $\overline{w}$. Note that $\overline{w}$ is nondecreasing, so has a unique shortest factorization in blocks of length $n_1, n_2, \dots, n_k$, with $n = n_1 + \cdots + n_k$, where each block is power of a single letter. Then

$$W_I = S_{n_1} \times \cdots \times S_{n_k}.$$

Following [2], we call *internal inversion* of w an inversion (i, j) such that for some $t \in \{0, \dots, k-1\}$, $n_1 + \cdots + n_t < i < j \leq n_1 + \cdots + n_{t+1}$; equivalently, the transposition (i, j) is in W_I, or $\overline{w}_i = \overline{w}_j$, or $w_{\sigma^{-1}(i)} = w_{\sigma^{-1}(j)}$. An inversion which is not internal will be called *external*.

THEOREM 4.1. *The mapping $w \mapsto w'$ preserves the number of internal and external inversions.*

PROOF. We refine the bijection, obtained in part 5 of the proof in Section 3, between inversions and couple of m-sequences.

Note that if (i, j) is an external inversion, then $w_{\sigma^{-1}(i)} \neq w_{\sigma^{-1}(j)}$. Hence the m-sequences obtained in the bijection are of length 2. Conversely, a couple of 2-sequences (a_1a_2, b_1b_2) with $a_1 < b_1$, $a_2 > b_2$ leads to an external inversion.
Thus, in the bijection:

- external inversions correspond to couple of 2-sequences;
- internal inversions correspond to couple of m-sequences with $m \geq 3$.

Since this is preserved under reversal, we obtain the theorem. □

Following [2], we say that a word is *minimal* if it has no internal inversions.

THEOREM 4.2. *Let w be a word, w' its inverse, σ, σ' the standard permutations of w, w'. Then the following conditions are equivalent:*

1. *w is minimal;*
2. *w' is minimal;*
3. *$\sigma' = \sigma^{-1}$;*
4. *$\sigma \in {}^I W \cap W^I$.*

Before proving the result, we note that one has always $\sigma \in {}^I W$. Indeed, it follows from the description of the Schensted standardization (see Section 2) that the number from 1 to n_1 appear in their natural order from left to right in σ, similarly for the number from $n_1 + 1$ to n_2, and so on (as previously, n_1 is the number of a terms in w, n_2 the number of b, etc...). This implies that the inverse descent set of σ, that is the descent set of σ^{-1}, is included in $\{1, \ldots, n\} \setminus I = \{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{k-1}\}$, hence that $\sigma^{-1} \in W^I$ and $\sigma \in {}^I W$.

PROOF. The equivalence of 1 and 2 follows from Theorem 4.1.

Suppose that 1 holds. Then w has no internal inversions, so that by the previous proof, there are no couple of m-sequences of $\Phi(w)$, $m \geq 3$, satisfying to the condition in 5, Section 3. This implies that for each couple of cycles $(a_1 \ldots a_k)$ and $(b_1 \ldots b_l)$ in $\Phi(w)$, one cannot have, for the lexicographical order of sequences: $(a_1, \ldots, a_k, a_1, \ldots, a_k, \ldots) < (b_1, \ldots, b_l, b_1, \ldots, b_l, \ldots)$ and $(a_1, a_k, \ldots, a_1, a_k, \ldots, a_1, \ldots) > (b_1, b_l, \ldots, b_1, b_l, \ldots, b_1, \ldots)$. So, in other words, ordering the sequences obtained by reading forward the circular words, or by reading them backwards, gives the same result. Thus, by the construction Φ^{-1} (see Section 2), we see that σ' is obtained by reversing the cycle of σ, i.e., $\sigma' = \sigma^{-1}$.

Suppose that 3 holds. Then, by the remark before the proof, we have $\sigma' = st(w') \in {}^I W$, hence $\sigma \in W^I$. Thus 4 holds.

Suppose that 4 holds. This implies that there is no internal descent in w (i.e., no $i \in I$ such that $w_i > w_{i+1}$), hence there can be no internal inversion and 1 holds. □

For each word w, there is a unique minimal word w_0 such that $w_0 \in wW_I$. We call it *the minimal word associated to w*. We call *parabolic standard permutation* of w the unique shortest permutation $\pi \in W_I$ such that $w = w_0\pi$. We denote $\pi = pst(w)$. As for standardization, parabolic standardization is easily described by a numbering process: one numbers separately the orbits under W_I.

THEOREM 4.3. *Let w be a word, $\sigma = st(w)$, $\pi = pst(w)$, w_0 the minimal word associated to w, $\sigma_0 = st(w_0)$, then*

1. *$\ell(\pi) =$ the number of internal inversions of w;*
2. *$\ell(\sigma_0) =$ the number of external inversions of w;*
3. *$\sigma = \sigma_0\pi$;*
4. *The inverse w'_0 of w_0 is the minimal word associated to w';*
5. *$\sigma' = st(w')$ is conjugate to σ^{-1} within W_I, that is: $\exists\alpha \in W_I$ such that $\sigma' = \alpha\sigma^{-1}\alpha^{-1}$.*

PROOF. By definition, 1 and 2 follow.

For 3 we have $w = w_0\pi = \overline{w}\sigma_0\pi$, and $l(\sigma_0\pi) \leq l(\sigma_0) + l(\pi) =$ number of inversions of $w = l(\sigma)$; since σ is shortest, we obtain $\sigma = \sigma_0\pi$.

For 4, note that a word and its associated minimal word have the same multiset of biletters. Moreover, a minimal word is completely determined by its set of biletters (see [2]). Since w_0 is minimal, w'_0 is minimal by Theorem 4.2. Then 4 follows from the previous remarks and Theorem 2.1.

For 5, take the notation of the end of Section 2. Let $M = \Phi(w)$, $E = \{u^\infty \mid (\widetilde{u}) \in M\}$. Similarly for w', M', E'. Since $M' = \mu(M)$ by definition, there is a canonical involutive bijection $\gamma : E \to E'$ that sends $(av)^\infty$ onto $(a\tilde{v})^\infty$. If s denotes the shift mapping, then $\gamma s = s^{-1}\gamma$, as is easily verified.

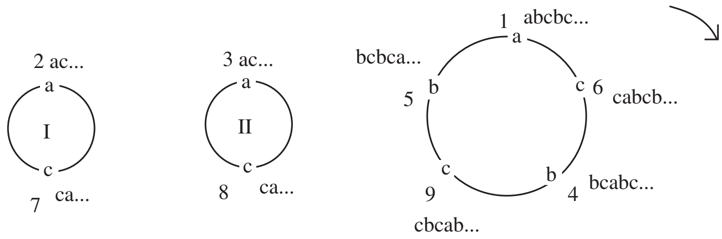

FIGURE 1. $\Phi(w)$.

Let $\beta : [n] \to E$ be as in Section 2, and similarly for β'. Then $\sigma = \beta^{-1}s^{-1}\beta$, $\sigma' = \beta'^{-1}s^{-1}\beta'$. Let $\alpha = \beta'^{-1}\gamma\beta$. Then $\alpha\sigma^{-1}\alpha^{-1} = \beta'^{-1}\gamma\beta\beta^{-1}s\beta\beta^{-1}\gamma\beta' = \beta'^{-1}\gamma s\gamma\beta' = \beta'^{-1}s^{-1}\beta' = \sigma'$.

We show that $\alpha \in W_I$. Equivalently, $\alpha^{-1} \in W_I$, which is equivalent to $\forall i$, $w_{\sigma^{-1}(i)} = w_{\sigma^{-1}\alpha^{-1}(i)}$. But $w_j = F\beta\sigma(j)$ (see Section 2), so that we must show that $F\beta\sigma\sigma^{-1}(i) = F\beta\sigma\sigma^{-1}\alpha^{-1}(i)$, that is $F\beta(i) = F\beta\beta^{-1}\gamma\beta'(i) = F\gamma\beta'(i)$. Since $F\gamma = F$, this is equivalent to $F\beta(i) = F\beta'(i)$. This is a consequence of the lexicographic ordering, since there are as many sequences beginning by a given letter in E as in E'. □

5. AN ALGORITHM

By Theorem 4.3, we know the existence of $\alpha \in W_I$ such that $\sigma' = \alpha\sigma^{-1}\alpha^{-1}$. Once α is known, so is σ', and we may obtain w' by $w' = \overline{w}\sigma'$. Here we compute α^{-1} directly from the set $I = \{(l, k) \mid l > k, \alpha(l) < \alpha(k)\}$, which is clearly in bijection with the set of inversions of α^{-1} and allows us classically to recover α^{-1}.

The algorithm computes I as follows: take any internal inversion (i, j) of w; then we have $i < j$, $\sigma(i) > \sigma(j)$ and $w_{\sigma^{-1}(i)} = w_{\sigma^{-1}(j)}$ (that is i, j are in the same orbit under W_I). Compute the couples $(\sigma^{-k}(j), \sigma^{-k}(i))$ for $k = 0, 1, 2, \ldots$ as long as $\sigma^{-k}(j)$ and $\sigma^{-k}(i))$ are in the same orbit under W_I. Then put these couples into I.

As an example, take $w = ccccabaab$. Then $\sigma = st(w) = 678914235$. The parabolic subgroup W_I is $S_3 \times S_2 \times S_4$; its orbits are $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7, 8, 9\}$; w and σ have three internal inversions: $(4, 5)$, $(6, 7)$, $(6, 8)$. We indicate by an arrow the diagonal action of σ^{-1} on the couples. We have $(5, 4) \to (9, 6) \to (4, 1)$ and the elements of the latter couple are not in the same orbit. Hence I contains $(5, 4)$ and $(9, 6)$; similarly, $(7, 6) \to (2, 1) \to (7, 5)$ and $(8, 6) \to (3, 1) \to (8, 5)$, so that $I = \{(5, 4), (9, 6), (7, 6), (2, 1), (8, 6), (3, 1)\}$, which shows that $\alpha^{-1} = 231547896$.

Thus we obtain $\sigma' = st(w') = \alpha\sigma^{-1}\alpha^{-1} = 674891253$. Hence $w' = \overline{w}\sigma' = aaabbccc.674891253 = ccbccaaba$.

In order to compare this construction of w' to the direct construction given in Section 2, we give the latter, with the notations of the proof of Theorem 4.3. We have, in cycle form, $\sigma = \{(16495), (27), (38)\}$; then, $M = \Phi(w) = \{(acbcb), (ac), (ac)\}$; recall that $E = \{(u)^\infty \mid (\tilde{u}) \in M\}$ and $E' = \{(u)^\infty \mid (u) \in M\}$; hence

$$E' = \{(acbcb)^\infty, (cbcba)^\infty, (bcbac)^\infty, (cbacb)^\infty, (bacbc)^\infty, (ac)^\infty, (ca)^\infty, (ac)^\infty, (ca)^\infty\}.$$

If we order lexicographically E', we obtain the bijection β' by listing the elements of E' from 1 to 9 (note how the multiplicities are taken into account): $E' = \{(ac)^\infty, (ac)^\infty,$

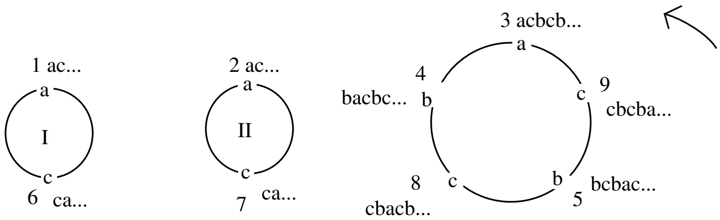

FIGURE 2. $\Phi(w')$.

$(acbcb)^\infty$, $(bacbc)^\infty$, $(bcbac)^\infty$, $(ca)^\infty$, $(ca)^\infty$, $(cbacb)^\infty$, $(cbcba)^\infty\}$. Therefore $\sigma' = \beta'^{-1}s^{-1}\beta' = 674891253$ and finally, since $w'_i = F\beta'\sigma'(i)$, w' is as above.

Note that the previous calculations have a graphical interpretation, which also gives immediately α: see Figure 1 and Figure 2.

6. COMPARISON WITH THE CONSTRUCTIONS OF FOATA–HAN AND CLARKE

In this section, we give two examples, one for Foata and Han's construction and the other for Clarke's construction, that show these constructions are not the same as ours. We do not give the constructions, just the results.

In [2], Foata and Han give the construction $w \mapsto w^*$. Their construction does not satisfy property 5 of Theorem 2.1. Following [2, p. 7], take the minimal word $w = c\,a\,b\,b$ (three inversions), then $w^* = b\,c\,b\,a$ (not a minimal word, four inversions).

To find w':

$$\begin{array}{ccccccc} w & = & c & a & b & b \\ \sigma & = & 4 & 1 & 2 & 3 \end{array}$$

then

$$\Phi\,(w) = \{(acbb)\}$$

Thus

$$\mu\Phi\,(w) = \Phi\,(w') = \{(abbc)\}.$$

Finally,

$$\begin{array}{cccccc} \sigma' & = & 2 & 3 & 4 & 1 \\ w' & = & b & b & c & a \end{array}$$

We see that $w' \neq w^*$.

Note, however, that for minimal words, our construction coincides with the one of Foata and Han; this follows from the fact that minimal words are completely determined by the multiset of their biletters. Moreover, minimal words are in one-to-one correspondence with certain matrices of natural integers (those whose first-row sum is equal to their first-column sum, similarly for the second and so on); as noted by Foata and Han, in this correspondence inversion corresponds to transposition of matrices, thus also to interchange tableaux in the generalization by Knuth [3] of the Schensted correspondence and of a theorem of Schützenberger [7].

In [1], Clarke gives the construction $w \mapsto w^{-1}$. His construction gives the good properties for three letters only. Following [1, Example 2], take the word $w = ccbccaccabbbbaa$, then $w^{-1} = ccbbcacccbbbaaa$.

To find w':

$$\begin{array}{ccccccccccccccccc} w & = & c & c & b & c & c & a & c & c & a & b & b & b & b & a & a \\ \sigma & = & 10 & 11 & 5 & 12 & 13 & 1 & 14 & 15 & 2 & 6 & 7 & 8 & 9 & 3 & 4 \end{array}$$

Then we have

$$\Phi(w) = \{(acb),\ (acbc),\ (abcbacbc)\} \quad \text{and} \quad \Phi(w') = \{(abc),\ (acbc),\ (acbcabcb)\}.$$

Finally,

$$\begin{array}{ccccccccccccccccc} \sigma' & = & 10 & 5 & 6 & 11 & 12 & 13 & 14 & 15 & 1 & 7 & 8 & 2 & 9 & 3 & 4 \\ w' & = & c & b & b & c & c & c & c & c & a & b & b & a & b & a & a \end{array}$$

We see that $w' \neq w^{-1}$.

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