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# On the Stern–Brocot expansion of real numbers

par CHRISTOPHE REUTENAUER

*In memoriam Michel Mendès-France*

RÉSUMÉ. Le développement de Stern–Brocot d’un nombre réel est une suite finie ou infinie de symboles  $l, r$ , signifiant « gauche » et « droite », qui représente le chemin dans l’arbre de Stern–Brocot déterminé par ce nombre. On montre que ce développement est périodique si et seulement si le nombre est quadratique, positif, avec conjugué négatif; dans ce cas la représentation de l’opposé du conjugué est obtenue par image miroir. Les pentes des suites sturmiennes morphiques sont exactement ces nombres. Deux nombres ont le même développement à partir d’un certain rang si et seulement s’ils sont équivalents sous l’action de  $SL_2(\mathbb{Z})$ . On obtient une relation d’adjacence pour les formes quadratiques binaires indéfinies, qui mène à un variante de la théorie des cycles de Gauss. Une bijection entre l’ensemble des mots de Lyndon sur deux lettres et les classes d’équivalence de ces formes est obtenue.

ABSTRACT. The Stern–Brocot expansion of a real number is a finite or infinite sequence of symbols  $r, l$ , meaning “right” and “left”, which represents the path in the Stern–Brocot tree determined by this number. It is shown that the expansion is periodic if and only if the number is positive quadratic with a negative conjugate; in this case the conjugate opposite’s expansion is obtained by reversal. The slopes of morphic Sturmian sequences are these quadratic numbers. Two numbers have ultimately the same expansion if and only they are  $SL_2(\mathbb{Z})$ -equivalent. A related neighbouring relation for indefinite binary quadratic forms leads to a variant of the Gauss theory of cycles. A bijection is obtained between the set of binary Lyndon words and  $SL_2(\mathbb{Z})$ -equivalence of these quadratic forms.

## 1. Introduction

The Stern–Brocot tree is an infinite complete planar binary tree, whose nodes are the positive rational numbers, see Figure 2.1. Each positive irrational real number  $\xi$  determines an infinite path in the tree, starting from the root. The path is coded by an infinite word on the alphabet  $l, r$  ( $l$  for left, and  $r$  for right). It is this infinite word that we call the *Stern–Brocot expansion* of  $\xi$ .

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We show that this expansion is periodic if and only if  $\xi$  is quadratic positive, with a negative conjugate  $\bar{\xi}$  (Theorem 3.1); one implication of this result was known, in an equivalent form, to Hurwitz; we give a new proof. A number satisfying the previous condition is called *good*. In this case, the Stern–Brocot expansion of  $-\bar{\xi}$  is obtained by reversing that of  $\xi$  (Corollary 3.8); this result is also due to Hurwitz, but we give a different proof.

We give a new proof of a theorem of Allauzen: a number is good if and only if it is the slope of some morphic Sturmian sequence (Theorem 5.1).

We consider *indefinite binary real quadratic forms*  $f(x, y) = ax^2 + bxy + cy^2$ :  $a, b, c \in \mathbb{R}$ ,  $b^2 - 4ac > 0$ ; we shall say “form” for short. Such a form is called *good* if  $a > 0, c < 0$ . We give a new proof of a result of Hurwitz: each form is equivalent, under the action of  $\mathrm{SL}_2(\mathbb{Z})$  on  $x, y$ , to a good form (Proposition 4.1). Then we show that in a given equivalence class of forms, the good forms are on a cycle  $(f_n)$ , with  $f_{n+1}(x, y) = f_n(x + y, y)$  or  $f_{n+1}(x, y) = f_n(x, x + y)$ ; the cycle is finite if and only if the form is proportional to a form with integral coefficients (Theorem 4.3 and Corollary 4.6). We obtain also an extension of a theorem of Lagrange: if for a form  $f$ , one has  $f(x, y) = m$  for some relatively prime integers  $x, y$ , then  $m$  is the first coefficient of some form in the cycle of the equivalence class of  $f$ .

We construct a natural bijection between the three following sets: the set of Lyndon words on the alphabet  $\{l, r\}$ ; the set of classes of real quadratic numbers under  $\mathrm{SL}_2(\mathbb{Z})$ -equivalence; the set of equivalence classes of forms (Theorem 6.1).

In the Appendix we review three results of Hurwitz, and give independent proofs of two of them. Two positive numbers have ultimately the same Stern–Brocot expansion if and only if they are equivalent under the action of  $\mathrm{SL}_2(\mathbb{Z})$  (Theorem A.1). If  $|\xi - \frac{p}{q}| < \frac{1}{q^2}$ , then  $\frac{p}{q}$  is a node on the Stern–Brocot path of  $\xi$ , with an improvement due to Fatou and Grace (Theorem A.3).

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## 2. Stern–Brocot expansion

**2.1. The results of Graham, Knuth and Patashnik.** In this section, we follow the nice book “Concrete Mathematics” of Graham, Knuth and Patashnik, [16, §4.5]. Let  $\xi$  be a positive irrational number. Let  $N(\xi) = \xi - 1$  if  $\xi > 1$ , and  $N(\xi) = \frac{\xi}{1-\xi}$  if  $\xi < 1$ . Note that  $N$  maps the set of positive irrational real numbers into itself.

Note also that the two cases are interchanged by inversion: if  $\xi > 1$ , then  $N(\xi^{-1}) = N(\xi)^{-1}$ .

Define the numbers  $\xi_n$  by  $\xi_0 = \xi$  and  $\xi_{n+1} = N(\xi_n)$ . The *Stern–Brocot expansion* of  $\xi$  is the infinite word  $a_0a_1a_2\cdots$  on the alphabet  $\{l, r\}$  such that  $a_n = r$  if  $\xi_n > 1$  and  $a_n = l$  if  $\xi_n < 1$ . For further use, note that the Stern–Brocot expansion of any number  $\xi_n$  is  $a_na_{n+1}a_{n+2}\cdots$ .

This infinite word is *not ultimately constant*, that is, it has  $l$ 's and  $r$ 's at arbitrary large ranks: indeed, suppose for example that  $a_n = r$  for  $n \geq n_0$ ; then  $\xi_n = \xi_{n_0} - (n - n_0)$ , which is negative for  $n$  large enough, a contradiction; similarly, by replacing numbers by their inverses, one sees that one cannot have  $a_n = l$  for  $n$  large enough.

All sequences in  $\{l, r\}^{\mathbb{N}}$  that are not ultimately constant are the Stern–Brocot expansion of some positive irrational number. This follows because the continued fraction of such a number is  $[n_0, n_1, n_2, \dots]$  if and only its Stern–Brocot expansion is  $r^{n_0}l^{n_1}r^{n_2}\dots$ , see [16, (6.140) p. 305]<sup>1</sup>.

To complete the picture, each rational number has a finite Stern–Brocot expansion: for some  $n$  the number  $\xi_n$  is equal to 1 and the process stops. Moreover, the correspondance between continued fractions and finite Stern–Brocot expansions is as follows: each positive rational number has exactly two continued fraction expansions  $[n_0, n_1, \dots, n_k + 1] = [n_0, n_1, \dots, n_k, 1]$  with  $k \geq 0$ , all  $n_i$  in  $\mathbb{N}$ , and  $n_i > 0$  for  $i = 1, \dots, k$ . Then the Stern–Brocot expansion of this number is  $r^{n_0}l^{n_1}r^{n_2}\dots(r \text{ or } l)^{n_k}$ .

Recall the left action of 2 by 2 matrices on complex numbers: for  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ,

$$M \cdot z = \frac{az + b}{cz + d}.$$

Define the homomorphism  $\mu$  from the free monoid  $\{l, r\}^*$  into  $SL_2(\mathbb{N})$  by

$$\mu(r) = R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mu(l) = L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

One has  $R^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ ,  $L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ . Hence  $\xi_{n+1} = R^{-1} \cdot \xi_n$  if  $\xi_n > 1$  and  $\xi_{n+1} = L^{-1} \cdot \xi_n$  if  $\xi_n < 1$ . Therefore  $\xi_n = R \cdot \xi_{n+1}$  if  $\xi_n > 1$  and  $\xi_n = L \cdot \xi_{n+1}$  if  $\xi_n < 1$ . Thus  $\xi_n = \mu(a_n) \cdot \xi_{n+1}$ . It follows that

$$\xi = \mu(a_0a_1 \cdots a_n) \cdot \xi_{n+1}.$$

The number  $\xi$  is recovered from its expansion as follows: let  $w_n = a_0a_1 \cdots a_{n-1}$ ,  $\mu(w_n) = \begin{pmatrix} b_n & c_n \\ d_n & e_n \end{pmatrix}$ ,  $r_n = \frac{b_n + c_n}{d_n + e_n}$ . Then

$$\xi = \lim_{n \rightarrow \infty} r_n.$$

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<sup>1</sup>This tight connection with continued fractions is perhaps the reason why the Stern–Brocot expansion has not been considered so often for itself.

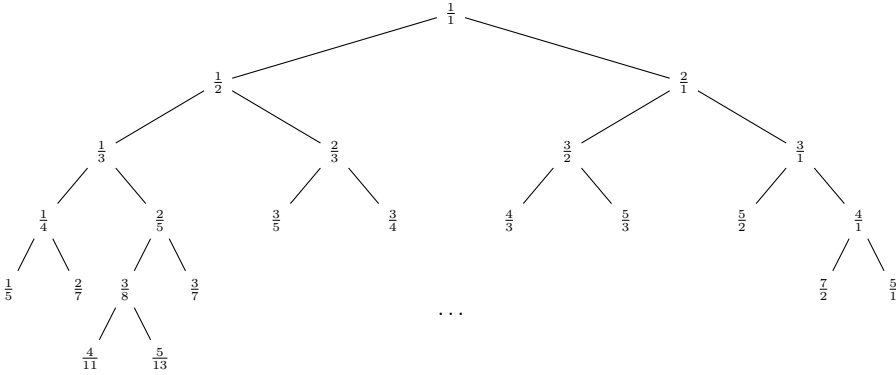


FIGURE 2.1. The Stern–Brocot tree

Note that this formula is a variant of Formula (4.34) in [16]: we have conjugated the matrices and therefore the numerator and denominator in the formula must be exchanged.

It follows also that the limit of  $\frac{b_n}{d_n}$  and  $\frac{c_n}{e_n}$  is also  $\xi$ . Indeed, since the determinant of  $\mu(w_n)$  is 1,

$$\frac{b_n}{d_n} - \frac{b_n + c_n}{d_n + e_n} = \frac{b_n d_n + b_n e_n - b_n d_n - c_n d_n}{d_n(d_n + e_n)} = \frac{1}{d_n(d_n + e_n)}$$

which tends to 0, since all coefficients of  $\mu(w_n)$  tend to  $\infty$  (because the  $a_n$  are not ultimately constant). Similarly for  $\frac{c_n}{e_n}$ .

In order to understand better the Stern–Brocot expansion, recall the *Stern–Brocot tree*. This is an infinite complete planar binary tree, whose nodes are the positive rational numbers. The root is 1. The nodes on the two extreme right and left branches are respectively  $1, 2, 3, 4, \dots$  and  $1, 1/2, 1/3, 1/4, \dots$  in this order. To compute the other nodes, consider such a node: the path originating at this node and towards the root has then right and left steps; take the first right step and the first left step, and the respective labels  $p_1/q_1, p_2/q_2$  of their target; then the label  $p/q$  of our node is the *mediant* of these two numbers, that is  $p = p_1 + p_2, q = q_1 + q_2$ . For example, see Figure 2.1:  $3/7$  is the mediant of  $1/2$  and  $2/5$ <sup>2</sup>.

<sup>2</sup>The lazy addition of fractions.

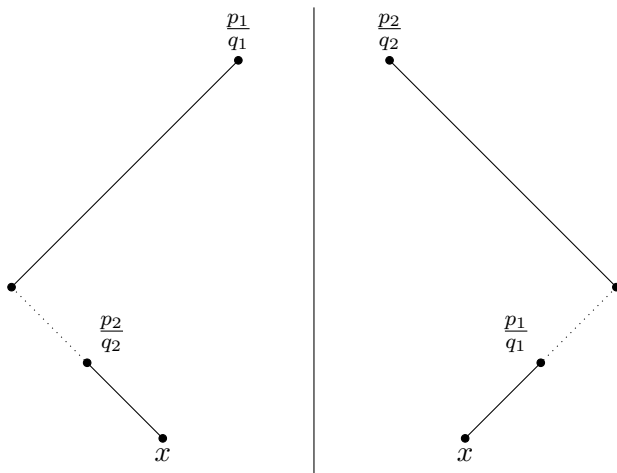


FIGURE 2.2. The two cases of the path from a node towards the root

More precisely, one has: let  $w \in \{l, r\}^*$  be the word coding the path from the root towards the node  $x$  considered (with  $l$  for “left” and  $r$  for “right”). Then

$$\mu(w) = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}.$$

See Figure 2.2, with the two cases. For example,  $x = 3/7$ ,  $w = llrr$  and  $\mu(w) = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ .

Now each positive rational number appears exactly once on the tree. Moreover, each positive irrational number  $\xi$  determines an infinite path on the tree as follows: it starts at the root 1, and if it passes through a node labelled  $p/q$ , it continues towards the left child if  $\xi < p/q$  and towards the right child if  $\xi > p/q$ .

It turns out that this infinite path, once coded by the letters  $l, r$ , is exactly the Stern–Brocot expansion of  $\xi$  as defined above.

In this path, all *convergents* of  $\xi$  appear: they are the label of the nodes which are followed by a node which is at a left or right turn on the path. The set of all labels on the path turns out to be the set of *semi-convergents* of  $\xi$ . For example, if the path is  $llrrrl \dots$ , the two first convergents are  $1/2$  and  $3/7$ , see Figure 2.1.

Recall that these numbers are defined as follows: suppose that the expansion of  $\xi$  (irrational) into continued fraction is  $\xi = [n_0, n_1, n_2, \dots]$ ,  $n_i \in \mathbb{Z}$ ,  $n_i > 0$  for  $i > 0$ . Then the convergents are all the rational numbers  $[n_0, \dots, n_k]$ , whereas the semi-convergents are all the numbers  $[n_0, \dots, n_{k-1}, j]$ ,  $k \geq 0$ ,  $j = 1, \dots, n_k$ .

Note that the Stern–Brocot tree has an alternative construction. Define the  $n$ -th *Stern–Brocot sequence*  $SB_n$  recursively by  $SB_{-1} = \frac{0}{1}, \frac{1}{0}$  and for  $n \geq -1$ ,  $SB_{n+1}$  is obtained by inserting their mediant between any two consecutive fractions in  $SB_n$ . Clearly,  $SB_n$  has  $2^{n+1} + 1$  terms. The first sequences ( $n = -1, 0, 1, 2, 3$ ) are

$\frac{0}{1}$									$\frac{1}{0}$							
$\frac{0}{1}$				$\frac{1}{1}$					$\frac{1}{0}$							
$\frac{0}{1}$		$\frac{1}{2}$		$\frac{1}{1}$		$\frac{2}{1}$			$\frac{1}{0}$							
$\frac{0}{1}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{2}{1}$	$\frac{3}{1}$		$\frac{1}{0}$							
$\frac{0}{1}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{3}{5}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{1}{1}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{2}{1}$	$\frac{5}{2}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{1}{0}$

The Stern–Brocot tree is then constructed in [16] as follows: the nodes at level  $n \geq 0$  are the elements of  $SB_n$  which are not in  $SB_{n-1}$ ; for example, the root is  $\frac{1}{1}$ , its left and right children are  $\frac{1}{2}$  and  $\frac{2}{1}$ , the left and right children of  $\frac{1}{2}$  are  $\frac{1}{3}$  and  $\frac{2}{3}$ , and so on.

For further use, we write

$$\xi = SB(s),$$

if  $s$  is a finite or infinite word on the alphabet  $\{l, r\}$  and  $\xi$  the real number having  $s$  as Stern–Brocot expansion.

**2.2. Historical remarks.** Constructions that are equivalent to the Stern–Brocot expansion have been considered previously, under other names.

In 1894, Hurwitz [17] defines the *Characteristik* of a real number; the construction uses the so-called Farey intervals; it is explained in the Appendix, with three theorems due to him.

The Stern–Brocot expansion has also been considered in 1973 by Raney [26]; he calls it *L, R-sequence expansion*. He constructs for each integral Möbius transformation a transducer that computes the expansion of the image of any real number under this transformation; motivation for this is to compute the continued fractions expansion of the image, knowing that one easily passes from the Stern–Brocot expansion to the continued fractions.

The Stern–Brocot expansion is also mentioned in [6, p. 36] under the name *Farey expansion*; the authors develop a similar theory of *sign sequences* for finite and infinite Christoffel words [6, p. 36] (the latter are in bijection with positive reals).

Finally, it is called *Stern–Brocot representation* by Niqui [24, p. 357] who extends Raney’s results to *quadratic maps*.

### 3. Periodic expansion

**Theorem 3.1.** *Let  $\xi$  be a positive irrational real number. Its Stern–Brocot expansion is periodic if and only if  $\xi$  is quadratic and if its conjugate  $\bar{\xi}$  is negative.*

This result is of course similar to the theorem of Galois asserting that a number  $\xi$  has a periodic expansion into continued fractions if and only if  $\xi$  is quadratic,  $\xi > 1$  and  $\bar{\xi} \in (-1, 0)$ . The condition for periodic Stern–Brocot expansions is even nicer, and the proof becomes very natural.

One implication of this result is equivalent to result of Hurwitz [18, Satz 24, p. 112]. The periodicity in his statement is somewhat hidden by the fact that the two first letters in the expansion are useless. The proof we give is self-contained and completely different of the proof by Hurwitz.

Note also that one could certainly deduce this theorem from results on Sturmian words; see Allauzen [2, Thm. 3] and [22, Exercise 2.3.4], attributed to Droubay, Justin, Pirillo.

Let us call *good* a number  $\xi$  that is positive, quadratic and such that its conjugate is negative; these numbers appear implicitly in Hurwitz’s article [18]. They appear also in [2]: Allauzen calls them the *Sturmian numbers of the first kind*, see [2, Thm. 3].

Denote  $\mathcal{A}$  the set of good numbers.

**Lemma 3.2.** *The mapping  $N$  is a bijection from  $\mathcal{A}$  into itself.*

*Proof.* Let  $\xi \in \mathcal{A}$ . Clearly,  $N(\xi)$  is positive and quadratic.

If  $\xi > 1$ , then  $\overline{N(\xi)} = \bar{\xi} - 1$  is negative, since  $\bar{\xi}$  is negative. Hence  $N(\xi) \in \mathcal{A}$ .

Observe that inversion maps  $\mathcal{A}$  into itself. Moreover  $N(\xi)^{-1} = N(\xi^{-1})$ . Hence if  $\xi < 1$ , the previous case shows that  $N(\xi) \in \mathcal{A}$  too.

Thus  $N$  maps  $\mathcal{A}$  into itself.

Let  $\nu = N(\xi)$ . We show that  $\xi > 1$  if and only if  $-\bar{\nu} > 1$ . This will imply that  $N$  is injective, since  $x - 1$  and  $\frac{x}{1-x}$  are both injective functions. Suppose indeed that  $\xi > 1$ ; then  $\nu = \xi - 1$ , hence  $\bar{\nu} = \bar{\xi} - 1 < -1$  (since  $\bar{\xi} < 0$ ), so that  $-\bar{\nu} > 1$ . Suppose now that  $\xi < 1$ ; then  $\xi^{-1} > 1$ , so that by what we have just shown,  $-\overline{N(\xi^{-1})} > 1$ ; since  $N(\xi^{-1}) = N(\xi)^{-1}$ , we obtain  $-\bar{\nu}^{-1} = -\overline{\nu^{-1}} = -\overline{N(\xi)^{-1}} = -\overline{N(\xi^{-1})} > 1$ , which implies that  $-\bar{\nu} = (-\bar{\nu}^{-1})^{-1} < 1$ .

To prove surjectivity, let  $\nu \in \mathcal{A}$ . If  $-\bar{\nu} > 1$ , let  $\xi = \nu + 1$ ; then  $\xi > 1 > 0$  (since  $\nu > 0$ ), and  $\bar{\xi} = \bar{\nu} + 1 < 0$ ; hence  $\xi \in \mathcal{A}$  and  $N(\xi) = \xi - 1 = \nu$ . Suppose now that  $-\bar{\nu} < 1$ ; then  $-\overline{\nu^{-1}} = (-\bar{\nu})^{-1} > 1$  (because  $\nu \in \mathcal{A} \Rightarrow \bar{\nu} < 0, -\bar{\nu} > 0$ ). Then applying what precedes to  $\nu^{-1} \in \mathcal{A}$ , we see that there exists  $\xi \in \mathcal{A}$  such that  $N(\xi) = \nu^{-1}$ ; we obtain that  $\xi^{-1} \in \mathcal{A}$  and  $N(\xi^{-1}) = \nu$ .  $\square$



**Lemma 3.3.** *Let  $\Delta$  be an integer. The set of triples  $(a, b, c)$  of integers, such that  $b^2 - 4ac = \Delta$  and that  $ac < 0$ , is finite.*

*Proof.* Under the assumptions, one has  $b^2 - 4ac = |b|^2 + 4|a||c|$ . Hence the number of triples  $(|a|, |b|, |c|)$  is finite. This implies the lemma.  $\square$

**Lemma 3.4.** *Suppose that irrational  $\xi$  satisfies an integer quadratic equation  $a\xi^2 + b\xi + c$  with discriminant  $b^2 - 4ac = \Delta$ . Let  $x = \xi - 1$  or  $x = \frac{\xi}{1-\xi}$ . Then  $x$  satisfies an integer quadratic equation  $a'x^2 + b'x + c' = 0$  with the same discriminant. Moreover, if  $ac < 0$ , and if either  $\xi > 1$  and  $x = \xi - 1$ , or  $0 < \xi < 1$  and  $x = \frac{\xi}{1-\xi}$ , then  $a'c' < 0$ .*

*Proof.* Suppose first that  $x = \xi - 1$ . Then  $0 = a(x+1)^2 + b(x+1) + c = ax^2 + (2a+b)x + a+b+c$ . The discriminant of this latter polynomial is  $(2a+b)^2 - 4a(a+b+c) = b^2 - 4ac$ . Suppose now that  $ac < 0$  and  $\xi > 1$ ; then the other root is  $< 0$  (because  $ac < 0$ ) and  $a, a+b+c$  have opposite sign (because 1 lies between the two roots). Since  $a' = a, c' = a+b+c$ , we obtain  $a'c' < 0$ .

Note that  $\xi^{-1}$  satisfies the quadratic equation with coefficients  $c, b, a$  with the same discriminant. Then the previous arguments applied to  $\xi^{-1}$  complete the proof, since  $x^{-1} = \xi^{-1} - 1$ .  $\square$

**Lemma 3.5.** *Let  $M$  be a 2 by 2 matrix whose entries are natural numbers. Suppose that  $M \notin \text{NI}_2$  and that  $M$  has a real irrational fixpoint  $\xi$  for the action of Section 2. Then  $M$  has a unique positive and a unique negative fixpoint. The positive one is a good number.*

*Proof.* Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then  $\xi = M \cdot \xi$  is equivalent to  $\xi = \frac{a\xi+b}{c\xi+d}$ , which may be rewritten  $c\xi^2 + (d-a)\xi - b = 0$ . Suppose that  $c = 0$ ; then  $(d-a)\xi - b = 0$  and, since  $\xi$  is irrational, we must have  $d = a$  and  $b = 0$ , a contradiction since  $M$  is not a scalar matrix. Similarly, if  $b = 0$ , then dividing by  $\xi$  (which is  $\neq 0$ ), we obtain  $c\xi + d - a = 0$ , and we reach a contradiction, too. Thus  $b, c$  are both nonzero and therefore the previous quadratic equation has two real solutions, and their product is negative. Hence the positive one is good.  $\square$

*Proof of Theorem 3.1.*

“ $\Rightarrow$ ”. Suppose that  $\xi$  is quadratic and that  $\bar{\xi}$  is negative. Then  $\xi$  satisfies an equation  $a\xi^2 + b\xi + c$ , as in Lemma 3.4; also,  $ac < 0$ , since  $\xi > 0$  and  $\bar{\xi} < 0$ . Define  $\xi_n$  as in Section 2: for each  $n$ ,  $\xi_{n+1} = N(\xi_n)$ . By Lemma 3.4. all the  $\xi_n$  satisfy such an equation, with the same discriminant and the condition on the two extreme coefficients. By Lemma 3.3, these equations are finitely many. Hence two of the  $\xi_n$  are equal:  $\xi_{n+p} = \xi_n$  for some positive  $p$ ; in other words,  $N^{n+p}(\xi) = N^n(\xi)$ . Since  $N$  is a bijection by Lemma 3.2, we deduce that  $N^p(\xi) = \xi$ . Thus the Stern–Brocot expansion of  $\xi$  is periodic.

“ $\Leftarrow$ ”. It follows from Section 2 that for any  $n$ ,  $\xi = M \cdot \xi_n$  for some  $M \in \text{SL}_2(\mathbb{N})$ , which is a product of  $L$  and  $R$ ’s; hence  $M$  has at least three nonzero coefficients and therefore  $M$  is not a scalar matrix.

Suppose now that the Stern–Brocot expansion of  $\xi$  is periodic. Then we have  $\xi = \xi_n$  for some  $n > 0$ . Therefore  $\xi = M \cdot \xi$ . We conclude with Lemma 3.5, since  $\xi$  is irrational.  $\square$

The following corollary is equivalent to the theorem of Lagrange characterizing ultimately periodic continued fractions. We give a proof since it is very short.

**Corollary 3.6.** *The Stern–Brocot expansion of an irrational real positive number  $\xi$  is ultimately periodic if and only if  $\xi$  is quadratic.*

*Proof.* If  $\xi$  has an ultimately periodic expansion, then  $\xi_n = \xi_p$  for some  $n < p$ . So the expansion of  $\xi_n$  is periodic and by the theorem  $\xi_n$  is quadratic. We have  $\xi = M \cdot \xi_n$  for some matrix  $M$  in  $\text{SL}_2(\mathbb{N})$ . Thus  $\xi$  is quadratic, too.

Conversely suppose that  $\xi$  is quadratic. Let  $a_0 a_1 a_2 \dots$  be its Stern–Brocot expansion and  $M_n = \mu(a_0 \dots a_{n-1})$ . We have  $\xi = M_n \cdot \xi_n$ . Hence  $\xi_n = M_n^{-1} \cdot \xi$  and therefore  $\bar{\xi}_n = M_n^{-1} \cdot \bar{\xi}$ . Let  $M_n = \begin{pmatrix} b_n & c_n \\ d_n & e_n \end{pmatrix}$ . We know that the limit of  $b_n/d_n$  and  $c_n/e_n$  is  $\xi$ . Since  $M_n^{-1} = \begin{pmatrix} e_n & -c_n \\ -d_n & b_n \end{pmatrix}$  we have

$$\bar{\xi}_n = \frac{e_n \bar{\xi} - c_n}{-d_n \bar{\xi} + b_n} = \frac{e_n}{-d_n} \frac{\bar{\xi} - \frac{c_n}{e_n}}{\bar{\xi} - \frac{b_n}{d_n}}.$$

The latter fraction tends to  $\frac{\bar{\xi} - \xi}{\bar{\xi} - \xi} = 1$ . Hence  $\bar{\xi}_n$  is negative for  $n$  large enough and  $\xi_n$  is good. Thus  $\xi_n$  has a periodic expansion by the theorem, and so  $\xi$  has an ultimately periodic expansion.  $\square$

**Corollary 3.7.** *Each real quadratic number is equivalent to a good number under the action of  $\text{SL}_2(\mathbb{Z})$ .*

*Proof.* This follows from the beginning of the previous proof if the number is positive. In general, any number is equivalent to a positive one, by adding a large enough integer  $n$ .  $\square$

We conclude this section by a result which is an analogue of a well-known result for continued fractions; it is due to Hurwitz ([18, Satz 4, p. 112]).

**Corollary 3.8.** *Suppose that  $\xi$  has a periodic Stern–Brocot expansion  $w^\infty$ ,  $w \in \{l, r\}^*$ . Let  $\bar{\xi}$  be its conjugate, as quadratic number. Then  $-\bar{\xi}$  has the expansion  $\tilde{w}^\infty$ , where  $\tilde{w}$  is the reversal of  $w$ .*

*Proof.* The real number  $\alpha > 0$  whose expansion is  $\tilde{w}^\infty$  satisfies  $\alpha = P \cdot \alpha$ , where  $P = \mu(\tilde{w})$ . We have to prove that  $\alpha = -\bar{\xi}$ . Let  $M = \mu(w)$ . Then  $P$  is the product of the matrices  $L$  and  $R$  in reverse order of the product giving  $M$ .

Consider the anti-automorphism  $\tau$  of 2 by 2 matrices which is transposition followed by conjugation by the permutation matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ . It sends each matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  onto  $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ . Therefore it fixes the matrices  $L$  and  $R$ . This shows that  $P = \tau(M)$ .

Let  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ ; it follows from the previous arguments that  $P = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ . Now  $\xi = M \cdot \xi$ , that is  $c\xi^2 + (d - a)\xi - b = 0$  and  $\alpha = P \cdot \alpha$ , that is  $c\alpha^2 + (a - d)\alpha - b = 0$ . Note that the roots of the first polynomial (that is,  $\xi$  and  $\bar{\xi}$ ) are the opposite of the roots of the second. Since each of these polynomials has roots of opposite sign, we deduce that the positive root of the second is the opposite of the negative root of the first. That is,  $\alpha = -\bar{\xi}$ .  $\square$

#### 4. Indefinite binary quadratic forms

**4.1. Good forms.** We consider *binary quadratic forms*  $f(x, y) = ax^2 + bxy + cy^2$ . We assume that they have real coefficients, and that they are *indefinite*: that is, their *discriminant*  $d(f) = b^2 - 4ac$  is positive. Call *roots* of  $f$  the roots of  $f(x, 1) = 0$ . We always *assume that the form has two irrational roots*. For short, we call this a *form*.

The group  $\text{GL}_2(\mathbb{Z})$  acts on the right on forms by  $(f \cdot M)(x, y) = f(px + qy, rx + sy)$ , when  $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ . Note that  $f(px + qy, rx + sy) = a(px + qy)^2 + b(px + qy)(rx + sy) + c(rx + sy)^2 = (ap^2 + bpr + cr^2)x^2 + (2apq + bps + bqr + 2crs)xy + (aq^2 + bqs + cs^2)y^2 = f(p, r)x^2 + (2apq + bps + bqr + 2crs)xy + f(q, s)y^2$ .

Two forms are (*properly*) *equivalent* if they are in the same orbit under the action of  $\text{SL}_2(\mathbb{Z})$ ; in the sequel, “equivalent” will mean “properly equivalent”. If the two forms lie in the same orbit under the action of  $\text{GL}_2(\mathbb{Z})$ , they are called *improperly equivalent*. Note that if  $f, g$  are properly or improperly equivalent, then they have the same discriminant (see [11, p. 66], [5, p. 4]).

We say that a form  $f(x, y) = ax^2 + bxy + cy^2$  is *good* if  $a > 0$  and  $c < 0$ . Note that then its roots have opposite sign:  $f$  has a *positive* and a *negative root*. This condition appears already in the article of Hurwitz [18, Satz 16, p. 100].

Call more generally positive (resp. negative) root of a (not necessarily good) form  $f(x, y) = ax^2 + bxy + cy^2$  the root  $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$  (resp.  $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ ); if  $f$  is good, since  $ac < 0$ , this coincides with the previous terminology.

**Proposition 4.1.** *Each form is equivalent to a good form.*

This result is due to Hurwitz ([18, Satz 17, p. 100]); we give a different proof.

**Lemma 4.2** ([11, Thm. 72 p. 99]).

- (i) Let  $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$  and let  $f, g$  be two quadratic forms such that  $f = g \cdot M$ . If  $u, v$  are the roots of  $f$ , then  $M \cdot u, M \cdot v$  are the roots of  $g$ .
- (ii) With the same notations, if  $M$  is in  $\text{SL}_2(\mathbb{Z})$  and if  $u$  is the positive root of  $f$ , then  $M \cdot u$  is the positive root of  $g$ .

Note that the denominators are nonzero, since  $u, v$  are irrational.

*Proof.*

(i). One has  $g(pu+q, ru+s) = (g \cdot M)(u, 1) = f(u, 1) = 0$ , hence  $g(\frac{pu+q}{ru+s}, 1) = 0$ . Thus  $M \cdot u$  is a root of  $g$ . Similarly for  $M \cdot v$ .

(ii). We show that if  $M \in \text{GL}_2(\mathbb{Z})$ ,  $f = g \cdot M$  and  $u$  is the positive root of  $f$ , then  $M \cdot u$  is the positive or negative root of  $g$ , depending if  $M$  has determinant 1 or  $-1$ . Since  $\text{GL}_2(\mathbb{Z})$  is generated by the two matrices  $R$  and  $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , it is enough to show this for these two matrices.

Let  $g = ax^2 + bxy + cy^2$ . For the first matrix, one has  $f(x, y) = g(x + y, y) = a(x + y)^2 + b(x + y)y + cy^2 = ax^2 + (2a + b)xy + (a + b + c)y^2$ ; then, letting  $d$  be the discriminant,  $u = \frac{-2a-b+\sqrt{d}}{2a}$  and  $M \cdot u = u + 1 = \frac{-b+\sqrt{d}}{2a}$ , the positive root of  $g$ .

For the second matrix,  $f(x, y) = g(y, x) = cx^2 + bxy + ay^2$ ,  $u = \frac{-b+\sqrt{d}}{2c}$  and  $M \cdot u = \frac{1}{u} = \frac{2c}{-b+\sqrt{d}} = \frac{2c(-b-\sqrt{d})}{(-b+\sqrt{d})(-b-\sqrt{d})} = \frac{2c(-b-\sqrt{d})}{b^2-b^2+4ac} = \frac{-b-\sqrt{d}}{2a}$ , the negative root of  $g$ . □

*Proof of Proposition 4.1.* Let  $f(x, y) = ax^2 + bxy + cy^2$  be a form, with  $\alpha, \beta$  its roots. If the roots have opposite sign, then  $ac < 0$  and the form is good, unless  $a < 0$  and  $c > 0$ ; but in this case,  $f(y, -x) = cx^2 - bxy + ay^2$  is equivalent to  $f$ , and good.

Suppose that the roots  $\alpha, \beta$  are both positive. They are distinct, so have two different Stern–Brocot expansions, say  $a_0 a_1 \dots, b_0 b_1 \dots$  respectively. We argue by induction on the rank of the first letter in these expansions that is different.

If they differ at rank 0, that is  $a_0 \neq b_0$ , then we may assume that  $\beta > 1$  and  $\alpha < 1$ . Then the form  $f(x + y, y)$  has roots  $\alpha - 1, \beta - 1$  and is equivalent to  $f$ ; since these numbers have opposite sign, we are done by the previous case.

Suppose that  $a_0 = r = b_0$ ; then  $\alpha, \beta > 1$ , hence the Stern–Brocot expansion of  $\alpha_1$  (resp.  $\beta_1$ ) is  $a_1 a_2 \dots$  (resp.  $b_1 b_2 \dots$ ). Then  $f(x + y, y)$  is equivalent to  $f$  and has roots  $\alpha_1 = \alpha - 1$  and  $\beta_1 = \beta - 1$ ; thus  $f(x + y, y)$  is by induction equivalent to a good form, and so is  $f$ . Suppose on the other hand that  $a_0 = l = b_0$ ; then  $f$  is equivalent to  $f(x, x + y)$ , whose roots are  $\alpha_1 = \frac{\alpha}{1-\alpha}$  and  $\beta_1 = \frac{\beta}{1-\beta}$ : indeed,  $f(\alpha_1, \alpha_1 + 1) = f(\frac{\alpha}{1-\alpha}, \frac{1}{1-\alpha}) = \frac{1}{1-\alpha} f(\alpha, 1) = 0$ ; the

Stern–Brocot expansion of  $\alpha_1$  (resp.  $\beta_1$ ) is  $a_1a_2\cdots$  (resp.  $b_1b_2\cdots$ ), so that  $f(x, x + y)$  is by induction equivalent to a good form, and so is  $f$ .

Suppose now that the roots are both negative. Then the roots of  $f(x - hy, y)$  are  $\alpha + h, \beta + h$ , which are positive for  $h$  large enough; since for  $h$  integer,  $f$  is equivalent to  $f(x - hy, y)$ , we are done by the previous cases.  $\square$

**4.2. Cycles of forms.** Let  $f(x, y) = ax^2 + bxy + cy^2$ . We write  $f \xrightarrow{r} g$  if  $a + b + c < 0$  and  $g(x, y) = f(x + y, y)$ , that is,  $g = f \cdot R$ . We write  $f \xrightarrow{l} g$  if  $a + b + c > 0$  and  $g(x, y) = f(x, x + y)$ , that is,  $g = f \cdot L$ . We write  $f \rightarrow g$  if  $f \xrightarrow{r} g$  or  $f \xrightarrow{l} g$  and say that  $g$  is a right neighbour of  $f$ ; we write  $f \xrightarrow{*} g$  if there is a  $\rightarrow$ -chain from  $f$  to  $g$ . Note that

$$(4.1) \quad \begin{aligned} f(x + y, y) &= ax^2 + (2a + b)xy + (a + b + c)y^2, \\ f(x, x + y) &= (a + b + c)x^2 + (b + 2c)xy + cy^2. \end{aligned}$$

It is useful to denote  $f = [a, b, c]$  for  $f(x, y) = ax^2 + bxy + cy^2$ . We then see that  $[a, b, c] \xrightarrow{r} [a, 2a + b, a + b + c]$  when  $a + b + c < 0$ , and  $[a, b, c] \xrightarrow{l} [a + b + c, b + 2c, c]$  when  $a + b + c > 0$ <sup>3</sup>.

Note also that one has never  $a + b + c = 0$ , since it is always assumed that  $f$  has only irrational roots; hence one has either  $f \xrightarrow{r} g$  or  $f \xrightarrow{l} g$ , for some  $g$ , and these cases are mutually exclusive.

**Theorem 4.3.**

- (i) If  $f$  is good and  $f \rightarrow g$ , then  $g$  is good.
- (ii) If  $g$  is good, there is a unique good  $f$  such that  $f \rightarrow g$ .
- (iii) If  $f, g$  are good and equivalent, then  $f \xrightarrow{*} g$  or  $g \xrightarrow{*} f$ .

We begin by prove a result of independent interest.

**Proposition 4.4.** *Suppose that  $f, g$  are good forms and that  $g = f \cdot M$  for some  $M \in \text{SL}_2(\mathbb{Z})$ . Then one of the four matrices  $M, M^{-1}, -M, -M^{-1}$  is in  $\text{SL}_2(\mathbb{N})$ .*

**Lemma 4.5.** *There exist no integers  $a, b, c, p, q, r, s$ , with  $qr - ps = 1$ , and such that  $a, p, q, r, s > 0, c < 0, ap^2 - bpr + cr^2 > 0$  and  $aq^2 - bqs + cs^2 < 0$ .*

*Proof.* Suppose the contrary. One has  $bpr < ap^2 + cr^2$  and  $bqs > aq^2 + cs^2$ . Hence  $\frac{aq^2 + cs^2}{qs} < b < \frac{ap^2 + cr^2}{pr}$ . It follows that  $aq^2pr + cs^2pr < ap^2qs + cr^2qs$ ; thus  $apq(qr - ps) < crs(qr - ps)$  and  $0 < apq < crs < 0$ , a contradiction.  $\square$

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<sup>3</sup>To memorize it in the case where  $f$  is good, recall that the first coefficient must be  $> 0$  and the last coefficient  $< 0$ .

*Proof of Proposition 4.4.* Suppose that the coefficients of  $M$  are all nonzero. Their sign defines a matrix  $S$  in  $\{+, -\}^{2 \times 2}$ , and the latter set has 16 elements. It is enough to show that  $S$  is one of the four matrices

$$\begin{pmatrix} + & + \\ + & + \end{pmatrix}, \begin{pmatrix} - & - \\ - & - \end{pmatrix}, \begin{pmatrix} + & - \\ - & + \end{pmatrix}, \begin{pmatrix} - & + \\ + & - \end{pmatrix},$$

since in the first (resp. second, resp. third, resp. fourth) case,  $M$  (resp.  $-M$ , resp.  $M^{-1}$ , resp.  $-M^{-1}$ ) is in  $\text{SL}_2(\mathbb{N})$ .

It is not possible that only one coefficient of  $M$  is negative; indeed, if for example,  $S = \begin{pmatrix} - & + \\ + & + \end{pmatrix}$ , then the determinant of  $M$  is  $\leq -2$ . The three other cases are similar, as are the cases where only one coefficient is positive.

There remain four cases; even only two, if one take opposites, noting that  $g = f \cdot M$  is equivalent to  $g = f \cdot (-M)$ . In these two cases,  $S$  is one of the two matrices

$$\begin{pmatrix} - & - \\ + & + \end{pmatrix}, \begin{pmatrix} - & + \\ - & + \end{pmatrix}.$$

Suppose that  $S$  is the first matrix. Then we may write  $M = \begin{pmatrix} -p & -q \\ r & s \end{pmatrix}$  with  $p, q, r, s > 0$ . Let  $f(x, y) = ax^2 + bxy + cy^2$ . Then, by a calculation made at the beginning of Subsection 4.1, the first and the last coefficients of  $g$  are  $f(-p, r) = ap^2 - bpr + cr^2$  and  $f(-q, s) = aq^2 - bqs + cs^2$  and we obtain a contradiction using Lemma 4.5.

Suppose that  $S$  is the second matrix. Then  $M = \begin{pmatrix} -p & q \\ -r & s \end{pmatrix}$ . The first and the last coefficients of  $g$  are  $ap^2 + bpr + cr^2$  and  $aq^2 + bqs + cs^2$  and we obtain a contradiction using the same lemma, with  $b$  replaced by  $-b$ .

The cases where one coefficient of  $M$  vanishes is treated as follows. Suppose that  $M = \begin{pmatrix} p & 0 \\ r & s \end{pmatrix}$  or  $M = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$ ; then, replacing  $M$  by its opposite if necessary, we may assume that  $p = s = 1$  and then it is readily seen that  $M$  or  $M^{-1}$  is in  $\text{SL}_2(\mathbb{N})$ .

If  $M = \begin{pmatrix} 0 & q \\ r & s \end{pmatrix}$ , then we see that the first coefficient of  $g$  is negative (it is  $cr^2$ ), a contradiction; and if  $M = \begin{pmatrix} p & q \\ r & 0 \end{pmatrix}$ , then the last coefficient of  $g$  is positive (it is  $aq^2$ ), a contradiction again.  $\square$

*Proof of Theorem 4.3.*

(i). This follows from (4.1).

(ii). We prove that  $f$  exists and is unique. Let  $g = ax^2 + bxy + cy^2$ ; then  $a > 0, c < 0$ . Then  $g \cdot R^{-1} = ax^2 + (-2a + b)xy + (a - b + c)y^2$  and  $g \cdot L^{-1} = (a - b + c)x^2 + (b - 2c)xy + cy^2$ . Since  $g$  has no rational root,  $a - b + c$  is nonzero and therefore exactly one of  $R^{-1} \cdot g$  or  $L^{-1} \cdot g$  is good.

(iii). This follows from the previous proposition since one has equivalence between the four equalities:  $g = f \cdot M$ ,  $g = f \cdot (-M)$ ,  $f = g \cdot M^{-1}$ ,  $f = g \cdot (-M^{-1})$ , since  $\text{SL}_2(\mathbb{N})$  is generated as monoid by  $R$  and  $L$ , and by the matrix definition of  $\rightarrow$ .  $\square$

Consider the directed graph whose vertices are the good forms and whose edges are  $f \rightarrow g$  with  $\rightarrow$  as previously defined. Thanks to the theorem, the restriction of this graph to the classes may be precisely described.

**Corollary 4.6.** *In each equivalence class of forms, the good forms are on an infinite or a finite cycle. The cycle is finite if and only if the forms in this class are proportional to an integral form.*

Hurwitz constructs a cycle quite differently; see [18, §7 p. 101–104]. I could not verify if his cycles coincide with the cycles constructed here; however, the very simple neighbouring relations  $f \rightarrow g$  seem not apparent in his article.

*Proof.* It follows from the theorem that for each equivalence class of forms, there exists good forms  $(f_n)_{n \in \mathbb{Z}}$  such that  $f_n \rightarrow f_{n+1}$  for any  $n$  and that the good forms in the class are exactly the  $f_n$ . If the  $f_n$  are distinct, the cycle is infinite.

If the  $f_n$  are not all distinct, then we have by the theorem

$$f_1 \rightarrow f_2 \rightarrow \dots \rightarrow f_p \rightarrow f_1,$$

for some  $p \geq 1$ . Thus we have  $f_1 = f_1.M$  for some matrix  $M \in \mathrm{SL}_2(\mathbb{N})$  and it follows then from Lemma 4.2(ii) that each root  $\xi$  of  $f_1$  satisfies  $\xi = \xi.M$ ; thus the two roots of  $f_1$  are conjugate quadratic numbers and it follows that  $f_1$  is proportional to the integral form  $ax^2 + bxy + cy^2$ , where  $ax^2 + bx + c$  is a nonzero integral polynomial having these two numbers as roots. It follows that each form in the cycle is proportional to an integral form.

Conversely, if  $f_1$  is proportional to an integral form, we may assume that  $f_1$  is integral; then, since the  $f_n$  have all the same discriminant, they are finitely many by Lemma 3.3, hence not distinct.  $\square$

As an example, consider the good form  $ax^2 + bxy + cy^2 = x^2 + xy - y^2$ , represented by  $[1, 1, -1]$ . Recall the rules before Theorem 4.3. Since here  $a + b + c > 0$ , we have  $[1, 1, -1] \xrightarrow{l} [1, -1, -1]$ ; now the sum of coefficients of the latter form is  $< 0$ , so that  $[1, -1, -1] \xrightarrow{r} [1, 1, -1]$ : the cycle is of length 2.

Take now the form  $[1, 2, -2]$ . We obtain

$$[1, 2, -2] \xrightarrow{l} [1, -2, -2] \xrightarrow{r} [1, 0, -3] \xrightarrow{r} [1, 2, -2],$$

and the cycle is of length 3 (in contrast, the Gauss cycles are always of even length, see [8, Prop. 3.6 p. 24]).

**Remark 4.7.** *The results of the present subsection and the previous one are similar to results of Gauss (see [5, Ch. 3] or [11, Ch. VII]); in the latter results one considers reduced forms (rather than good forms as here). An indefinite binary quadratic form  $ax^2 + bxy + cy^2$  is reduced if*

$0 < \sqrt{d(f)} - b < 2|a| < \sqrt{d(f)} + b$ . The neighbouring relations  $f \rightarrow g$  in the Gauss cycles are somewhat more complicated to compute (see [11, p. 102] or [5, p. 23]).

Another notion of reduction, leading to similar results, is given by Zagier [30, p. 122]: he calls reduced a form such that  $a > 0, c > 0, b > a + c$ .

**4.3. Representation of a number by a form.** We say that a form  $f(x, y)$  represents *primitively* (or *properly*) a number  $m$  if for some relatively prime integers  $x, y$  one has  $f(x, y) = m$ .

The following theorem is an extension of a theorem of Lagrange, see [11, Thm. 85 p. 111]<sup>4</sup>.

**Theorem 4.8.** *Suppose that the form  $f$  represents primitively the positive number  $m$  with  $m < \sqrt{d(f)}$ . Then  $m$  is the first coefficient of some good form equivalent to  $f$ ; equivalently it is the first coefficient of a form in the cycle associated to  $f$ .*

It is remarkable that the factor  $\frac{1}{2}$ , necessary in Lagrange’s theorem, is not necessary in the present result; this is similar of what happens for quadratic equations and generalized Pell equations, as noted by Hurwitz [17, §4 p. 425-427]; see also Subsection A.4.

*Proof.* It is well-known that, since  $f$  represents primitively  $m$ , there is an equivalent form  $f(x, y) = ax^2 + bxy + cy^2$  such that  $a = m$ . This is seen as follows: we have  $f(p, r) = m$  for some relatively prime integers  $p, r$ ; there exist integers  $q, s$  such that  $ps - qr = 1$ ; the matrix  $M = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$  is in  $SL_2(\mathbb{Z})$ ; then, by a calculation made in Subsection 4.1,  $f.M = f(p, r)x^2 + (\dots)xy + f(q, s)y^2$ .

Thus we may assume that  $f(x, y) = ax^2 + bxy + cy^2$  and  $a = m$ . The point of the proof is that the hypothesis  $a < \sqrt{d(f)}$  implies that the difference between the two roots of  $f$ , namely  $\frac{-b \pm \sqrt{d(f)}}{2a}$ , is  $> 1$ , hence there is some integer  $h$  lying strictly between them and therefore  $f(h, 1) < 0$  (because  $a > 0$ ). Now the form  $f(x + hy, y) = f.\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$  is equivalent to  $f$ , and we may conclude since it is equal to  $ax^2 + (\dots)xy + f(h, 1)y^2$ , which is a good form. □

### 5. Morphic Sturmian sequences

We begin by recalling some facts on Sturmian words. Details may be found in Lothaire’s book [22, Ch. 2], [3, Ch. 9 and 10] or [14, Ch. 6]. A *Sturmian word* is a non ultimately periodic infinite word  $s$  on the alphabet  $\{a, b\}$  which satisfies the following *balance property*: for any factors  $u, v$  of

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<sup>4</sup>This theorem asserts that if a quadratic form  $f$  represents primitively an integer  $m$ , satisfying  $|m| < \frac{1}{2}d(f)$ , then  $m$  is the first coefficient of some quadratic form in the Gauss cycle determined by  $f$ .



equal length of  $s$ , one has  $|u|_a - |v|_a = -1, 0$  or  $1$  (here  $|w|_a$  denotes the number of  $a$ 's in  $w$ )

It is then shown that the limit, when the length of  $w$  tends to infinity, of the number  $\frac{|w|_b}{|w|_a}$ , for the factors  $w$  of  $s$ , exists. We call this number the *slope* of  $s$ .

For a proof of the existence of the limit, see [22, Prop. 2.1.7]. Note that there is considered the quantity  $\frac{|w|_b}{|w|_a + |w|_b}$ , but this does not change the existence of the limit.

Our slope of a Sturmian word, which is in the spirit of the article [6] of Borel and Laubie, is not the same than that of [22], which follows the definition of Morse and Hedlund [23] (it is the limit of  $\frac{|w|_b}{|w|}$ ). The former behaves better with respect to Sturmian morphisms than the latter, as we see below.

A *Sturmian morphism* is a monoid homomorphism from  $\{a, b\}^*$  into itself, which is *nonerasing* (that is,  $f(a), f(b)$  nonempty), and such that it preserves Sturmian sequences.

If  $f$  is a homomorphism from  $\{a, b\}^*$  into itself, its *abelianization*  $M(f)$  is the matrix

$$M(f) = \begin{pmatrix} |f(b)|_b & |f(a)|_b \\ |f(b)|_a & |f(a)|_a \end{pmatrix}.$$

It is well-known that one has the following matrix relation, for any word  $w$ :

$$(5.1) \quad \begin{pmatrix} |f(w)|_b \\ |f(w)|_a \end{pmatrix} = M(f) \begin{pmatrix} |w|_b \\ |w|_a \end{pmatrix}$$

Note that this implies that

$$\frac{|f(w)|_b}{|f(w)|_a} = M(f) \cdot \frac{|w|_b}{|w|_a},$$

using the action of matrices on numbers recalled in Section 2: indeed, letting  $M(f) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ , we have  $|f(w)|_b = p|w|_b + q|w|_a$ ,  $|f(w)|_a = r|w|_b + s|w|_a$ , and

$$\frac{|f(w)|_b}{|f(w)|_a} = \frac{p|w|_b + q|w|_a}{r|w|_b + s|w|_a} = \frac{p\frac{|w|_b}{|w|_a} + q}{r\frac{|w|_b}{|w|_a} + s} = M(f) \cdot \frac{|w|_b}{|w|_a}.$$

It follows from this equality that if  $s = f(t)$  for two Sturmian sequences  $s, t$ , of respective slopes  $\alpha, \beta$ , then ([6, Thm. 4])

$$(5.2) \quad \alpha = M(f) \cdot \beta,$$

if we assume that  $f$  is nonerasing. Indeed, let  $w_n$ ,  $n \in \mathbb{N}$ , be factors of increasing length of  $t$ . Then  $\frac{|w_n|_b}{|w_n|_a}$  tends to  $\beta$  when  $n$  tends to infinity.

Similarly,  $\frac{|f(w_n)|_b}{|f(w_n)|_a}$  tends to  $\alpha$ , since the length of  $f(w_n)$  tends to infinity.

Now, we have

$$\frac{|f(w_n)|_b}{|f(w_n)|_a} = M(f) \cdot \frac{|w_n|_b}{|w_n|_a}.$$

By taking the limit, we obtain (5.2).

A Sturmian word is called *morphic* if it is the fixpoint of some nonerasing endomorphism of the free monoid<sup>5</sup>, which is not the identity.

**Theorem 5.1.** *A number is good if and only if it is the slope of a morphic Sturmian sequence.*

This theorem is due to Allauzen (see [2, Def. p. 238, Thms. 1 and 3]). It is also related to Theorem 12 of [12] (see also [20, Thm. 3.6]), which characterizes morphic (strict standard) Sturmian sequences by the periodicity of their *directive sequence*. Earlier work on slopes of morphic Sturmian sequences has been done in [7, Thms. 1,2,3,4], [19, Thm. p. 288], [10, Thm. 2 and 3], [5, Cor. p. 48], [4, Thm. 3.7], [21, Cor. p. 352], [25, Thm. 3]; see also [27].

*Proof.* The “if” part follows from (5.2) and Lemma 3.5, since the slope of a Sturmian sequence is irrational ([22, Prop. 2.1.11 and Thm. 2.1.13]), and since the abelianization of the morphism  $f$  is not a scalar matrix: indeed, if it were such a matrix, then the morphism would be of the form  $f(a) = a^n, f(b) = b^n$  for some  $n \geq 2$  ( $n = 1$  is impossible since  $f$  is not the identity); then since the Sturmian word  $s$  contains both the letters  $a$  and  $b$  (it is not periodic),  $s = f(s)$  would contain the factors  $aa$  and  $bb$ , contradicting the balance property.

For the converse, we use the proof of Theorem 3.1: we have  $\xi = M \cdot \xi$ , where  $M$  is a product of matrices  $L$  and  $R$ ; this product has  $L$ ’s and  $R$ ’s in it. We take the corresponding product  $f$  of morphisms ( $a \mapsto a, b \mapsto ab$ ) and ( $a \mapsto ab, b \mapsto b$ ), whose abelianizations are  $R$  and  $L$ . Then  $f$  sends  $a$  onto  $au, u \neq 1$ , and therefore  $f$  has a fixpoint  $s$ , which is the limit of the finite words  $f^n(a)$  (each of which is a proper prefix of the next one). Then by (5.2) the slope of  $s$  is  $\xi$ . The infinite word  $s$  is Sturmian, since the two previous morphisms are Sturmian, so that  $f^n(a)$  is balanced and since the slope of  $s$  is irrational, see [22, Prop. 2.1.11 and Thm. 2.1.5].  $\square$

## 6. Binary Lyndon words, classes of good numbers and quadratic forms

A finite word is *primitive* if it is not a nontrivial power of another one. It is well-known that each periodic infinite word is of the form  $w^\infty$  for some unique primitive word  $w$ . Order the alphabet  $\{l, r\}$  by  $l < r$  and extend it to a lexicographical order on the free monoid  $\{l, r\}^*$ ; a *Lyndon word* is

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<sup>5</sup>An important result of Berstel and Séebold ([4, Thm. 3.2]; see also [22, Thm. 2.3.7]) implies that the morphism must then be Sturmian, but we do not need this here.

a primitive word that is the smallest, for the lexicographical order, in its conjugation class (that is the words obtained by circular permutations of this word). For example,  $lrlrr$  is a Lyndon word, being smaller than the words  $rlrrl, lrllr, rrlrl, rlrllr$ .

An integral quadratic form is called *primitive* if its coefficients are relatively prime. Then each form in its  $SL_2(\mathbb{Z})$ -equivalence class is primitive.

**Theorem 6.1.** *There are natural bijections between*

- (i) *the set of Lyndon words on  $\{l, r\}$ , excluding  $l$  and  $r$ ;*
- (ii) *the set of classes of real quadratic numbers under  $SL_2(\mathbb{Z})$ -equivalence;*
- (iii) *the set of primitive indefinite integral binary quadratic forms under  $SL_2(\mathbb{Z})$ -equivalence.*

Note that a counting argument by Uludağ, Zeytin and Durmuş predicts the existence of a bijection between (i) and (iii) (see [29, p. 455–456]). In the proof below, we construct an explicit one.

**Lemma 6.2.** *Let  $\xi$  be a good number and  $\xi_n = N^n(\xi)$ . Let  $d_0d_1d_2\cdots$  be the Stern–Brocot expansion of  $\xi$ . Let  $a_n, b_n, c_n$  be relatively prime integers, with  $a_n > 0$ , such that  $\xi_n$  is a root of  $a_nx^2 + b_nx + c_n$ . Let  $f_n(x, y) = a_nx^2 + b_nxy + c_ny^2$ . Then  $f_n \xrightarrow{d_n} f_{n+1}$ .*

*Proof.* Observe that  $a_nx^2 + b_nx + c_n$  is the unique minimal polynomial of  $\xi_n$  whose coefficients are relatively prime integers and whose dominant coefficient is positive. Note that all the  $\xi_n$  are good numbers (Lemma 3.2) and all the quadratic forms  $f_n$  are good forms.

Suppose that  $\xi_n > 1$ ; then  $d_n = r$  and  $\xi_{n+1} = \xi_n - 1$ . Since  $\bar{\xi}_n < 0$ ,  $a_n > 0$  and thus  $1 \in [\bar{\xi}_n, \xi_n]$ , we have  $a_n + b_n + c_n < 0$ . Hence  $f_n \xrightarrow{r} g$  for some good form  $g$  (Theorem 4.3 (i)). It is enough to show that  $g = f_{n+1}$ . By (4.1),  $g = [a_n, 2a_n + b_n, a_n + b_n + c_n]$  and  $\xi_{n+1}$  is a root of  $a_n(x+1)^2 + b_n(x+1) + c_n = a_nx^2 + (2a_n + b_n)x + a_n + b_n + c_n$ . The coefficients of this latter polynomial are relatively prime and its dominant coefficient is positive. Hence  $g = f_{n+1}$  by the previous observation.

Suppose that  $\xi_n < 1$ ; then  $d_n = l$  and  $\xi_{n+1} = \frac{\xi_n}{1 - \xi_n}$ , equivalently  $\xi_{n+1}^{-1} = \xi_n^{-1} - 1$ . Since  $\bar{\xi}_n < 0$  and thus  $1 \notin [\bar{\xi}_n, \xi_n]$ , we have  $a_n + b_n + c_n > 0$ . Hence  $f_n \xrightarrow{l} g$  for some good form  $g$ . It is enough to show that  $g = f_{n+1}$ . By (4.1),  $g = [a_n + b_n + c_n, b_n + 2c_n, c_n]$  and, since  $\xi_n^{-1}$  is a root of  $c_nx^2 + b_nx + a_n$ ,  $\xi_{n+1}^{-1}$  is a root of  $c_n(x+1)^2 + b_n(x+1) + a_n$ , which is equal to  $c_nx^2 + (2c_n + b_n)x + c_n + b_n + a_n$ . Hence  $\xi_{n+1}$  is a root of  $(a_n + b_n + c_n)x^2 + (b_n + 2c_n)x + c_n$ , which implies as above that  $g = f_{n+1}$ .  $\square$

*Proof of Theorem 6.1.* Associate with each Lyndon word  $u$  the  $SL_2(\mathbb{Z})$ -class of the real number whose Stern–Brocot expansion is  $u^\infty$ . This number is quadratic by Theorem 3.

The previous mapping is an injective mapping: indeed, if  $u, v$  are Lyndon words such that the numbers with expansion  $u^\infty, v^\infty$  are  $SL_2(\mathbb{Z})$ -equivalent, then by a theorem of Hurwitz (Theorem A.1 in the Appendix), these two infinite words are ultimately equal, hence equal since they are periodic; then one must have  $u = v$ , because they are Lyndon words.

The mapping is surjective: let  $\xi$  be any real quadratic number; up to equivalence, we may assume that  $\xi > 0$  (since  $\xi$  is equivalent to any  $\xi + n$ ,  $n \in \mathbb{N}$ ). Its Stern–Brocot expansion is ultimately periodic (Corollary 3.6), hence of the form  $uv^\infty$  for some Lyndon word  $v$  and some word  $u$ ; then the expansion of  $\xi_{|u|}$  is  $v^\infty$ , and we know that this latter number is equivalent to  $\xi$ . Then, by definition of the mapping, the class of  $\xi$  is in the image of  $v$ .

Thus we have obtained a bijection between the sets (i) and (i).

We give now a bijection between the sets (ii) and (iii). With any real quadratic number, associate first an equivalent good number  $\xi$  (Corollary 3.7). Then associate with  $\xi$  the class of the form  $f = ax^2 + bxy + cy^2$ , where  $ax^2 + bx + c$  is the unique integral polynomial having this number as root, with  $a > 0$  and  $a, b, c$  relatively prime;  $f$  is good since  $\bar{\xi} < 0$ , hence  $c < 0$ , and  $\xi$  is necessarily the positive root of  $f$ .

We verify that this mapping is well-defined. If  $\tau$  is a good number equivalent to  $\xi$  under  $SL_2(\mathbb{Z})$ , then by Theorem A.1 their Stern–Brocot expansions (which are periodic by Theorem 3.1) are shift each of another. It follows then from Lemma 6.2 that the corresponding forms are equivalent. Hence, the mapping  $\xi \mapsto \text{class of } f$  is well-defined.

The mapping is injective: let indeed  $\xi, \alpha$  be two quadratic numbers having the same image under the mapping. Take two good numbers  $\xi', \tau'$  respectively equivalent to them. Then, by the previous construction of the mapping, their images are the classes of good forms whose positive roots are respectively  $\xi', \tau'$ ; since by hypothesis these forms are equivalent, it follows from Lemma 4.2 that  $\xi', \tau'$  are equivalent. Hence so are  $\xi, \tau$ .

Surjectivity follows from the fact that each class  $C$  of forms contains a good form  $f$ . Then  $C$  is the image of the equivalence class of the positive root of  $f$ . □

Given a Lyndon word  $w$ , the corresponding good form is obtained as follows: let  $\mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ ; then the form is  $\frac{1}{d}(rx^2 + (s - p)xy - qy^2)$ , where  $d = \gcd(s - p, q, r) > 0$ . Indeed, the number  $\xi$  whose Stern–Brocot expansion is  $w^\infty$  satisfies  $\xi = \mu(w) \cdot \xi = \frac{p\xi + q}{r\xi + s}$ , thus  $r\xi^2 + (s - p)\xi - q = 0$ .

As a byproduct, we obtain for each class of forms a canonical representative (compare to [29, p. 456]). It is characterized by the fact that its positive root is the smallest one in the cycle; this follows since comparing positive real numbers amounts to compare lexicographically their Stern–Brocot expansions. It would be interesting to characterize this unique representative intrinsically by its coefficients.

The bijection between classes of forms and Lyndon words may be recovered directly from the cycle of good forms in the equivalence class of a given form. We give first a property of this cycle. We use the fact that a primitive good form is completely determined by its positive root.

**Proposition 6.3.** *Let  $f$  a good form and suppose that the cycle of  $f$  is of the form  $f_i \xrightarrow{a_{i+1}} f_{i+1}$  with  $f_0 = f, \dots, f_N = f, a_1 \cdots a_N = w^n$  for some primitive word  $w$  and some integer  $n \geq 1$ . Then  $f_{|w|} = f$ .*

**Lemma 6.4.** *Suppose that  $\xi, \alpha$  are positive real numbers and that  $\xi = M \cdot \alpha$  for some matrix  $M$  in  $SL_2(\mathbb{N})$ . Then for some  $n \in \mathbb{N}$  we have  $\alpha = \xi_n$ . In particular, with  $w \in \{l, r\}^*$  defined by  $\mu(w) = M$ , the Stern–Brocot expansion of  $\xi$  is the concatenation of the word  $w$  and of the Stern–Brocot expansion of  $\alpha$ .*

*Proof.* Recall that  $SL_2(\mathbb{N})$  is a monoid freely generated by the matrices  $R$  and  $L$ . We argue by the length of  $M$  in this free monoid. If the length is 0, there is nothing to prove. Suppose that the length is  $> 0$ . Then either  $M = PR$  or  $M = PL$ , with a shorter  $P$  in the monoid. Let  $\beta = R \cdot \alpha$  in the first case and  $\beta = L \cdot \alpha$  in the second case. Then  $\beta > 0$ . Moreover  $\xi = P \cdot \beta$ , so by induction we obtain that  $\beta = \xi_n$  for some  $n$ .

In the first case, we have  $\beta = \alpha + 1$ , so that  $\beta > 1$ ; hence  $N(\beta) = \beta - 1$  and  $\alpha = R^{-1} \cdot \beta = \beta - 1 = N(\beta) = N(\xi_n) = \xi_{n+1}$ .

In the second case,  $\beta = \frac{\alpha}{\alpha+1}$ , so that  $\beta < 1$ ; hence  $N(\beta) = \frac{\beta}{-\beta+1}$  and  $\alpha = L^{-1} \cdot \beta = N(\beta) = N(\xi_n) = \xi_{n+1}$ . □

*Proof of Proposition 6.3.* We may assume that  $f$  is primitive. Let  $M = \mu(w)$ . Then  $f = f \cdot M^n$ . By Lemma 4.2, the positive root  $\xi$  of  $f$  satisfies  $\xi = M^n \cdot \xi$ . Hence its Stern–Brocot expansion is  $w^\infty$ , by Lemma 6.4. Hence  $\xi = M \cdot \xi$  and  $\xi = M^{-1} \cdot \xi$ . Let  $g = f.M$  so that  $g$  is on the cycle, hence good, and  $f = g.M^{-1}$ . It follows from Lemma 4.2 that  $g$  has  $M^{-1} \cdot \xi = \xi$  as root. Thus,  $g$  having  $\xi$  as root, being good and primitive, must be equal to  $f$ . Thus  $f = f.M$  and the lemma follows. □

**Corollary 6.5.** *The bijection from (iii) to (i) in the Theorem 6.1 is computed directly as follows: for a given class, the associated Lyndon word is the label of a cycle, which up to conjugation, is a Lyndon word.*

*Proof.* Indeed, let the cycle be  $f_0 \rightarrow f_1 \cdots \rightarrow f_n = f_0$ . Then by the proposition, the word  $w \in \{l, r\}^*$  labelling the cycle is primitive. We may suppose that it is a Lyndon word. It is enough to show that in the bijection from (i) to (iii),  $w$  is mapped onto the class of  $f_0$ . We have  $f_0 \cdot \mu(w) = f_0$ . It follows from Lemma 4.2 that the positive root of  $f_0$  satisfies the quadratic equation  $r\xi^2 + (s-p)\xi - q = 0$ , where  $\mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ . Thus  $f_0 = \frac{1}{d}(rx^2 + (s-p)xy - qy^2)$ , where  $d = gcd(s - p, q, r) > 0$ , since  $f_0$  is good and primitive. It follows

that  $w$  is mapped onto the class of  $f_0$ , as was seen after the proof of Theorem 6.1. □

Let us see some examples. As follows from the examples at the end of Section 4, the Lyndon word  $lr$  corresponds to the class of the quadratic form  $[1, 1, -1]$ , while the Lyndon word  $lrr$  correspond to the class of the form  $[1, 2, -2]$ . The corresponding classes of numbers are those of the positive roots of theses forms, that is  $\frac{\sqrt{5}-1}{2}$  and  $\sqrt{3} - 1$  respectively.

**Appendix: Three theorems of Hurwitz**

**A.1. The construction of Hurwitz.** First, we review how Hurwitz defines in [17] the expansion of an irrational number  $\xi$ . Recall that the Farey sequence  $\mathcal{F}_n$  of order  $n$  is defined as follows:  $\mathcal{F}_{-1} = \frac{0}{1}, \frac{1}{0}$ , and for  $n \geq -1$ ,  $\mathcal{F}_{n+1}$  obtained from  $\mathcal{F}_n$  by inserting between any two consecutive fractions  $\frac{r}{u}, \frac{s}{v}$  of  $\mathcal{F}_n$ , such that  $u + v, r + s \leq n + 2$ , their median  $\frac{r+s}{u+v}$  ([17, Satz 2 p. 420])<sup>6</sup>. For example, the first Farey sequences  $\mathcal{F}_n, n = -1, 0, 1, 2, 3$ , are

$$\begin{array}{ccccccccc}
 \frac{0}{1} & & & & & & & & & \frac{1}{0} \\
 \\
 \frac{0}{1} & & & & \frac{1}{1} & & & & & \frac{1}{0} \\
 \\
 \frac{0}{1} & & \frac{1}{2} & & \frac{1}{1} & & \frac{2}{1} & & & \frac{1}{0} \\
 \\
 \frac{0}{1} & & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & & \frac{1}{1} & & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & & \frac{1}{0} \\
 \\
 \frac{0}{1} & \frac{1}{4} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \frac{1}{1} & \frac{4}{3} & \frac{3}{2} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{1}{0}
 \end{array}$$

Comparing with Section 2, it is readily seen that the  $n$ -th Farey sequence is a subsequence of the  $n$ -th Stern–Brocot sequence.

If  $x$  is an irrational number, and  $n$  some natural integer, then  $x$  lies in some interval  $[\frac{r}{u}, \frac{s}{v}]$ , where  $\frac{r}{u}, \frac{s}{v}$  are two consecutive members of the Farey sequence  $\mathcal{F}_n$ . This pair is called a *pair of approximating fractions* of  $x$  (“Paar von Näherungsbrüchen”, [17, p. 423–424]). With  $x$  is then associated the decreasing sequence of all such intervals. If  $[\frac{r}{u}, \frac{s}{v}]$  is an interval in this sequence, then the next one is of the form  $[\frac{r}{u}, \frac{r+s}{u+v}]$  or  $[\frac{r+s}{u+v}, \frac{s}{v}]$ ; in other words, only one bound of the two intervals is new at each step ([17, p. 424]).

Consider the sequence of these new bounds. It may be shown that this sequence of fractions is the set of nodes of the path defined by  $x$  in the Stern–Brocot tree, as in Section 2. Thus the Stern–Brocot expansion is obtained by replacing each fraction by  $r$  if  $x$  is larger than the fraction, and by  $l$  if it is smaller. But this is exactly what does Hurwitz in order to

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<sup>6</sup>Note that we have shifted the indices. Moreover we have considered only the positive part of the sequences in [17]. Note also that classical Farey sequences are restricted to numbers between 0 and 1

obtain the *characteristic* (“Characteristik”) of  $x$ , except that he uses  $-$  and  $+$  instead of  $r$  and  $l$ , see [17, p. 428]<sup>7</sup>.

Let us take the example of Hurwitz, with  $x = \sqrt{2}$ . The first nine intervals are

$$\left[\frac{0}{1}, \frac{1}{0}\right], \left[\frac{1}{1}, \frac{1}{0}\right], \left[\frac{1}{1}, \frac{2}{1}\right], \left[\frac{1}{1}, \frac{3}{2}\right], \left[\frac{4}{3}, \frac{3}{2}\right], \left[\frac{7}{5}, \frac{3}{2}\right], \left[\frac{7}{5}, \frac{10}{7}\right], \left[\frac{7}{5}, \frac{17}{12}\right], \left[\frac{24}{17}, \frac{17}{12}\right], \dots$$

The sequence of new bounds is

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{10}{7}, \frac{17}{12}, \frac{24}{17}, \dots$$

The characteristic is therefore  $- + + - - + + \dots$  and the Stern–Brocot sequence is  $rllrrll \dots$ .

It is shown by Hurwitz ([17, Satz 4 p. 424]) that if  $x \in [\frac{r}{u}, \frac{s}{v}]$  and  $us - vr = 1$ , then  $\frac{r}{u}, \frac{s}{v}$  is a pair of approximating fractions of  $x$ . It follows from Section 2 that these two fractions are semi-convergents of  $x$ .

**A.2. Numbers equivalent under  $\mathrm{SL}_2(\mathbb{Z})$ .** It is well-known that two irrational reals have ultimately the same expansion in continued fractions if and only if they lie in the same orbit under the action of  $\mathrm{GL}_2(\mathbb{Z})$  (*Serret’s theorem*).

What happens for the smaller orbits under  $\mathrm{SL}_2(\mathbb{Z})$ ? This is the following result of Hurwitz.

**Theorem A.1** (Hurwitz [17, statement p. 434]). *Two positive numbers have ultimately the same Stern–Brocot expansion if and only if they lie in the same orbit under  $\mathrm{SL}_2(\mathbb{Z})$ .*

Let  $\xi$  be a positive real irrational number. Recall that the numbers  $\xi_n$  have been defined in Section 2:  $\xi_n = N^n(\xi)$ . Recall that the monoid  $\mathrm{SL}_2(\mathbb{N})$  is generated freely by the matrices  $L$  and  $R$ . In other words, the homomorphism  $\mu$  is a bijection  $\{l, r\}^* \rightarrow \mathrm{SL}_2(\mathbb{N})$ .

*Proof.* If  $\xi$  and  $\alpha$  have ultimately the same Stern–Brocot expansion, then  $\xi_n = \alpha_p$  for some  $n, p$ . Since  $\xi = M \cdot \xi_n$  and  $\alpha = P \cdot \alpha_p$  for some matrices  $M, P$  in  $\mathrm{SL}_2(\mathbb{N})$ , we obtain that  $\alpha = PM^{-1} \cdot \xi$ , hence  $\xi$  and  $\alpha$  are in the same orbit under  $\mathrm{SL}_2(\mathbb{Z})$ .

Conversely, suppose that  $\alpha, \xi$  are positive irrationals such that  $\alpha = Q \cdot \xi$  for some matrix  $Q \in \mathrm{SL}_2(\mathbb{Z})$ . We may write  $\xi = M_n \cdot \xi_n$  for some matrix  $M_n$  which is a product of matrices  $L$  and  $R$ . Note that the coefficients of  $M_n$  tend to infinity with  $n$ , since the product involves both  $L$  and  $R$ ’s. Write

$$Q = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, M_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

<sup>7</sup>He says that he follows Christoffel [9, p. 259]; the word “Characteristik” used by Christoffel refers however to something else: it is, in more recent terms, the Sturmian sequence associated to  $x$ .

We may assume that  $r\xi + s > 0$ , replacing  $Q$  by  $-Q$  if necessary. We have  $\alpha = (QM_n) \cdot \xi_n$ , so that

$$\alpha = \frac{A_n \xi_n + B_n}{C_n \xi_n + D_n},$$

with  $QM_n = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}$ ,  $C_n = ra_n + sc_n$ ,  $D_n = rb_n + sd_n$ . We know that the limit of  $\frac{a_n}{c_n}$  is  $\xi$ , and similarly for  $\frac{b_n}{d_n}$ . Hence  $a_n = \xi c_n + \epsilon_n c_n$  and  $b_n = \xi d_n + \epsilon'_n d_n$  where  $\epsilon_n$  and  $\epsilon'_n$  tend to 0. We have  $C_n = (r\xi + s + r\epsilon_n)c_n$ ,  $D_n = (r\xi + s + r\epsilon'_n)d_n$ . Thus for  $n$  large enough,  $C_n, D_n > 0$ . Now the matrix  $QM_n$  is in  $SL_2(\mathbb{Z})$ , has positive coefficients in its second row. The coefficients in its first row are then either both nonnegative, or both nonpositive (otherwise its determinant is  $\geq 2$  or  $\leq -2$ ). The second case cannot occur since  $0 < \alpha = (QM_n) \cdot \xi_n$  and  $\xi_n > 0$ . Thus  $QM_n \in SL_2(\mathbb{N})$ .

We now apply Lemma 6.4 to the equality  $\alpha = (QM_n) \cdot \xi_n$  and conclude that for some  $p$ ,  $\alpha_p = \xi_n$ . Hence the expansions of  $\xi$  and  $\alpha$  are ultimately equal. □

By Serret’s theorem and by the correpondance between continued fractions and Stern–Brocot representation (see Section 2), the corollary below follows.

**Corollary A.2.** *Two irrational numbers are  $GL_2(\mathbb{Z})$ -equivalent if and only if their Stern–Brocot expansions are ultimately equal, up to the exchange of  $l$  and  $r$ .*

Each number is  $GL_2(\mathbb{Z})$ -equivalent to its inverse. The number  $\sqrt{3}$  is not  $SL_2(\mathbb{Z})$ -equivalent to its inverse; indeed the Stern–Brocot expansion of  $\sqrt{3}$  and its inverse are respectively  $(rlr)^\infty$  (because the expansion of  $\sqrt{3} - 1$  is  $(lrr)^\infty$ , see the end of Section 6) and  $(lrl)^\infty$  (as indicates the computation  $\xi = \frac{1}{\sqrt{3}} < 1$ ,  $\xi_1 = \frac{\sqrt{3}+1}{2} > 1$ ,  $\xi_2 = \frac{\sqrt{3}-1}{2} < 1$ ,  $\xi_3 = \xi$ ).

**A.3. Quadratic error implies semi-convergent.** Recall that, by a theorem of Legendre, for each real number  $\xi$ , each rational number  $p/q$  satisfying  $|\xi - p/q| < 1/2q^2$  must be a convergent of  $\xi$ . The converse is however not true, although each convergent satisfies the inequality  $|\xi - p/q| < 1/q^2$ . But the converse of this latter statement is not true either: there are rational numbers satisfying this equality and which are not convergents. However, Hurwitz proved the following result, improved by Fatou and Grace<sup>8</sup>.

**Theorem A.3.** *If  $|\xi - p/q| < 1/q^2$ , then:*

- (1) (Hurwitz [17, Satz 5 p. 424])  $p/q$  is a semi-convergent of  $\xi$ ;
- (2) (Fatou [13, p. 1019], [15, p. 259–250])  $p/q$  is a convergent or adjacent to a convergent.

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<sup>8</sup>I am indebted to Yann Bugeaud for indicating me the results of Fatou and Grace and the references.



In other words,  $p/q$  is the label of some node on the path of the Stern–Brocot tree determined by  $\xi$ , and this node corresponds to a convergent (see the description of these nodes in Section 2.1), or is adjacent to such a node.

The proof of the theorem in is two parts. For the first assertion, we give a variant of the argument of Hurwitz; for the second, we follow an idea of Grace, using the following lemma, of independent interest, and which is similar to a calculation of the error term for convergents, see e.g. [1, (1.15) p. 23].

**Lemma A.4.** *Let  $\xi = \text{SB}(s)$ ,  $s \in \{l, r\}^\infty$ ,  $s = wxs'$ ,  $w \in \{l, r\}^*$ ,  $x \in \{l, r\}$ , and  $p/q = \text{SB}(w)$ . Then*

$$\xi - \frac{p}{q} = \frac{1}{\lambda q^2},$$

with two cases:

- (a)  $x = l$ ,  $-\lambda = \text{SB}(s') + q'/q$ ,  $p'/q'$  is the node after the first left step on the path from  $p/q$  towards the root on the Stern–Brocot tree;
- (b)  $x = r$ ,  $\lambda = \text{SB}(s')^{-1} + q'/q$ ,  $p'/q'$  is the node after the first right step on this path.

*Proof.* Suppose that we are in the first case. We use several times the basic results of Section 2.1. We have  $\mu(wl) = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$  (see Figure 2.2 right part). We have  $\xi = \mu(wl) \cdot \xi'$  with  $\xi' = \text{SB}(s')$ . Thus  $\xi - \frac{p}{q} = \frac{p\xi' + p'}{q\xi' + q'} - \frac{p}{q} = \frac{-1}{q(q\xi' + q')} = \frac{-1}{q^2(\xi' + q'/q)}$ . Thus  $-\lambda = \xi' + q'/q$ .

In the second case, we have  $\mu(wr) = \begin{pmatrix} p' & p \\ q' & q \end{pmatrix}$ . Then, with  $\xi' = \text{SB}(s')$ ,  $\xi = \mu(wr) \cdot \xi' = \frac{p'\xi' + p}{q'\xi' + q} = \frac{p' + p\xi'^{-1}}{q' + q\xi'^{-1}}$ ; hence  $\xi - \frac{p}{q} = \frac{p' + p\xi'^{-1}}{q' + q\xi'^{-1}} - \frac{p}{q} = \frac{1}{q(q' + q\xi'^{-1})} = \frac{1}{q^2(\xi'^{-1} + q'/q)}$ . Thus  $\lambda = \xi'^{-1} + q'/q$ . □

*Proof of Theorem A.3.* (1). The number  $p/q$  appears on the Stern–Brocot tree. Hence it is the mediant of two earlier nodes  $p_1/q_1 < p_2/q_2$ :  $p = p_1 + p_2$ ,  $q = q_1 + q_2$ . Note that  $p_2q_1 - p_1q_2 = 1$  (see Section 2); it follows that  $pq_1 - p_1q = (p_1 + p_2)q_1 - p_1(q_1 + q_2) = 1$  and similarly  $p_2q - p_1q_2 = 1$ . Thus  $p/q - p_1/q_1 = 1/q_1q \geq 1/q^2$  and  $p_2/q_2 - p/q = 1/q_2q \geq 1/q^2$ .

It follows from the hypothesis that  $|\xi - p/q| < p/q - p_1/q_1, p_2/q_2 - p/q$ . Thus  $\xi$  is in one of two intervals  $[p_1/q_1, p/q]$  or  $[p/q, p_2/q_2]$ . It follows from Subsection A.1 that  $p/q$  is a semi-convergent of  $\xi$ .

(2). By (1), we may write  $\xi = \text{SB}(s)$ ,  $s = w \cdots$ ,  $p/q = \text{SB}(w)$ ,  $s \in \{l, r\}^\infty, w \in \{l, r\}^*$ . Suppose that  $p/q$  is not a convergent, nor a semi-convergent adjacent to some convergent. Suppose that we are in case (a). of the lemma. Then, using the geometric description of the convergents of  $\xi$  given in Section 2.1, we must have  $s = wlll \cdots$ : otherwise  $s = wlr$  or  $wllr$

and then  $w$  is a convergent or adjacent to a convergent. Then  $s' = ll \cdots$  and therefore  $\xi' < 1/2$ . Moreover, we must have  $w = url^i$  and  $i \geq 1$ : otherwise,  $w = ur$  and since  $s = url \cdots$ ,  $u$  is a convergent and  $w$  is adjacent to a convergent. Let  $\mu(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . We have  $\mu(wl) = \mu(u)RL^{i+1}$  and thus with the notations of the proof of the lemma  $q' = c + d, q = (i + 2)c + (i + 1)d$  so that  $q'/q < 1/2$ . It follows that  $|\lambda| < 1$ , and we cannot have  $|\xi - p/q| < 1/q^2$ .

The case (b) of the lemma is similar. □

The converse of the theorem is not true, as shows any  $\xi = \text{SB}(rrl \cdots)$ ; then  $\text{SB}(r) = 2$  is a convergent, so that  $1/1$  (having empty Stern–Brocot expansion) is a semi-convergent adjacent to a convergent; one has  $\xi - 1 = \xi_1 = \text{SB}(rl \cdots) > 1/1^2$ .

To see some examples, let  $\xi = 1 + \sqrt{2}$ . Its partial quotients are all equal to 2. The convergents are  $2, 5/2, 12/5, \dots$ . The other semi-convergents are  $1, 3, 7/3, 17/7, \dots$ . One may verify that  $3, 7/3, 17/7$  satisfy the hypothesis of the previous theorem. Note that the convergents are exactly the fractions  $P_{n+1}/P_n$  and the other semi-convergents are  $1$  and the fractions  $(P_{n+1} + P_n)/(P_n + P_{n-1})$ ,  $n \geq 0$ , where  $P_n$  are the Pell numbers defined by  $P_{-1} = 0, P_0 = 1$  and  $P_{n+1} = 2P_n + P_{n-1}$  for  $n \geq 0$ .

#### A.4. Primitive representation of numbers.

**Theorem A.5** (Hurwitz [17, first statement p. 427]). *Let  $f(x, y) = ax^2 + bxy + cy^2$  be an indefinite binary quadratic form with  $a$  or  $c > 0$ . Suppose that  $u, v$  are relatively prime integers such that  $au^2 + buv + cv^2 = m$  and  $0 < m < \sqrt{d(f)}$ <sup>9</sup>. Then  $\frac{u}{v}$  is a semi-convergent of one of the roots of  $f$ .*

Note that one uses here also negative semi-convergents, by extending the Farey sequences to negative numbers. This result (and the proof) is similar to a result that one may find in the book of Serret [28, Thm. p. 80]: the stronger inequality there is  $m < \frac{1}{2}\sqrt{d(f)}$  and the stronger conclusion is that  $\frac{u}{v}$  is a convergent of one of the roots of  $f$ .

### References

- [1] M. AIGNER, *Markov’s Theorem and 100 years of the Uniqueness Conjecture, a mathematical journey from irrational numbers to perfect matchings*, Springer, 2013.
- [2] C. ALLAUZEN, “Une caractérisation simple des nombres de Sturm”, *J. Théor. Nombres Bordeaux* **10** (1998), no. 2, p. 237-241.
- [3] J.-P. ALLOUCHE & J. SHALLIT, *Automatic sequences. Theory, applications, generalizations*, Cambridge University Press, 2003.
- [4] J. BERSTEL & P. SÉÉBOLD, “Morphismes de Sturm”, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), no. 2, p. 175-189.
- [5] J.-P. BOREL, “Image par homographie de mots de Christoffel”, *Bull. Belg. Math. Soc. Simon Stevin* **8** (2001), no. 2, p. 241-255.

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<sup>9</sup>Note that the factor 2 in Hurwitz’s statement is explained by the fact that he considers quadratic forms of the form  $ax^2 + 2bxy + cy^2$  and  $d(f) = b^2 - ac$ .

- [6] J.-P. BOREL & F. LAUBIE, “Quelques mots sur la droite projective réelle”, *J. Théor. Nombres Bordeaux* **5** (1993), no. 1, p. 23-51.
- [7] T. C. BROWN, “A characterization of the quadratic irrationals”, *Can. Math. Bull.* **34** (1991), no. 1, p. 36-41.
- [8] D. A. BUELL, *Binary quadratic forms. Classical theory and modern computations*, Springer, 1989.
- [9] E. B. CHRISTOFFEL, “Lehrsätze über arithmetische Eigenschaften der Irrationalzahlen”, *Annali di Mat.* **XV** (1887), p. 253-276.
- [10] D. CRISP, W. MORAN, A. POLLINGTON & P. SHIUE, “Substitution invariant cutting sequences”, *J. Théor. Nombres Bordeaux* **5** (1993), no. 1, p. 123-137.
- [11] L. E. DICKSON, *Introduction to the theory of numbers*, Dover Publications, 1957.
- [12] X. DROUBAY, J. JUSTIN & G. PIRILLO, “Episturmian words and some constructions of de Luca and Rauzy”, *Theor. Comput. Sci.* **255** (2001), no. 1-2, p. 539-553.
- [13] P. FATOU, “Sur l’approximation des incommensurables et les séries trigonométriques”, *C. R. Math. Acad. Sci. Paris* **139** (1904), p. 1019-1021.
- [14] N. P. FOGG, V. BERTHÉ, S. FERENCZI, C. MAUDUIT & A. SIEGEL (eds.), *Substitutions in Dynamics, Arithmetic and Combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer, 2002.
- [15] J. H. GRACE, “The classification of rational approximations”, *Proc. Lond. Math. Soc.* **17** (1918), p. 247-258.
- [16] R. L. GRAHAM, D. E. KNUTH & O. PATASHNIK, *Concrete mathematics*, Addison-Wesley Publishing Group, 1994.
- [17] A. HURWITZ, “Ueber die angenäherte Darstellung der Zahlen durch rationale Brüche”, *Math. Ann.* **44** (1894), p. 417-436.
- [18] ———, “Ueber die Reduction der binären quadratischen Formen”, *Math. Ann.* **45** (1894), p. 85-117.
- [19] S. ITO & S.-I. YASUTOMI, “On continued fractions, substitutions and characteristic sequences  $[nx + y] - [(n-1)x + y]$ ”, *Jap. J. Math.* **16** (1990), p. 287-306.
- [20] J. JUSTIN & G. PIRILLO, “Episturmian words and episturmian morphisms”, *Theor. Comput. Sci.* **276** (2002), no. 1-2, p. 281-313.
- [21] T. KOMATSU & A. J. VAN DER POORTEN, “Substitution invariant Beatty sequences”, *Jap. J. Math.* **22** (1996), no. 2, p. 349-354.
- [22] M. LOTHAIRE, *Algebraic combinatorics on words*, Encyclopedia of Mathematics and Its Applications, vol. 90, Cambridge University Press, 2002.
- [23] M. MORSE & G. A. HEDLUND, “Symbolic dynamics II: Sturmian Trajectories”, *Am. J. Math.* **62** (1940), p. 1-42.
- [24] M. NIQUI, “Exact arithmetic on the Stern–Brocot tree”, *J. Discrete Algorithms* **5** (2007), no. 2, p. 356-379.
- [25] B. PARVAIX, “Propriétés d’invariance des mots sturmiens”, *J. Théor. Nombres Bordeaux* **9** (1997), no. 2, p. 351-369.
- [26] G. N. RANEY, “On continued fractions and finite automata”, *Math. Ann.* **206** (1973), p. 265-283.
- [27] P. SÉÉBOLD, “Sturmian images of non Sturmian words and standard morphisms”, *Theor. Comput. Sci.* **711** (2018), p. 92-104.
- [28] J.-A. SERRET, *Cours d’algèbre supérieure. I.*, éditions Jacques Gabay, 1992.
- [29] A. M. ULUDAĞ, A. ZEYTIN & M. DURMUŞ, “Binary quadratic forms as dessins”, *J. Théor. Nombres Bordeaux* **29** (2017), no. 2, p. 445-469.
- [30] D. B. ZAGIER, *Zetafunktionen und quadratische Körper, Eine Einführung in die Zahlentheorie*, Springer, 1981.

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