

Counting the Ideals of Given Codimension of the Algebra of Laurent Polynomials in Two Variables

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ABSTRACT. We establish an explicit formula for the number $C_n(q)$ of ideals of codimension (colength) n of the algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of Laurent polynomials in two variables over a finite field \mathbb{F}_q of cardinality q . This number is a palindromic polynomial of degree $2n$ in q . Moreover, $C_n(q) = (q - 1)^2 P_n(q)$, where $P_n(q)$ is another palindromic polynomial; the latter is a q -analogue of the sum of divisors of n , which happens to be the number of subgroups of \mathbb{Z}^2 of index n .

1. Introduction

Let \mathbb{F}_q be a finite field of cardinality q , and let $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ be the algebra of Laurent polynomials in two variables with coefficients in \mathbb{F}_q .

Our main aim is to give a formula for the number $C_n(q)$ of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. By codimension of an ideal I we mean the dimension of the quotient vector space $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]/I$ over \mathbb{F}_q .

Our main result is the following.

THEOREM 1.1. *For each integer $n \geq 1$, we have*

$$C_n(q) = \sum_{\lambda \vdash n} (q - 1)^{2v(\lambda)} q^{n-\ell(\lambda)} \prod_{\substack{i=1, \dots, t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1},$$

where the sum runs over all partitions λ of n . The expression $C_n(q)$ is a monic polynomial of degree $2n$ in the variable q with integer coefficients. Moreover, the polynomial $C_n(q)$ is divisible by $(q - 1)^2$.

The notation $\ell(\lambda)$, $v(\lambda)$, d_i appearing in the formula will be explained in Section 3.1. The proof of the theorem will be given in Section 5.3; it relies on a parameterization by Conca and Valla [8] of the affine cells in the Ellingsrud–Strømme decomposition of the Hilbert scheme of n points on the affine plane.

Note that since $C_n(q)$ is divisible by $(q - 1)^2$, for each $n \geq 1$, we may define a unique polynomial $P_n(q)$ by

$$C_n(q) = (q - 1)^2 P_n(q), \tag{1.1}$$

which clearly implies $C_n(1) = 0$ for all $n \geq 1$. Table 1 (resp. Table 2) displays the polynomials $C_n(q)$ (resp. the polynomials $P_n(q)$) for $n \leq 12$.

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Table 1 The polynomials $C_n(q)$

n	$C_n(q)$
1	$q^2 - 2q + 1$
2	$q^4 - q^3 - q + 1$
3	$q^6 - q^5 - q^4 + 2q^3 - q^2 - q + 1$
4	$q^8 - q^7 - q + 1$
5	$q^{10} - q^9 - q^7 + q^6 + q^4 - q^3 - q + 1$
6	$q^{12} - q^{11} + q^7 - 2q^6 + q^5 - q + 1$
7	$q^{14} - q^{13} - q^{10} + q^9 + q^5 - q^4 - q + 1$
8	$q^{16} - q^{15} - q + 1$
9	$q^{18} - q^{17} - q^{13} + q^{12} + q^{11} - q^{10} - q^8 + q^7 + q^6 - q^5 - q + 1$
10	$q^{20} - q^{19} - q^{11} + 2q^{10} - q^9 - q + 1$
11	$q^{22} - q^{21} - q^{16} + q^{15} + q^7 - q^6 - q + 1$
12	$q^{24} - q^{23} + q^{15} - q^{14} - q^{10} + q^9 - q + 1$

Table 2 The polynomials $P_n(q)$

n	$P_n(q)$	$P_n(1)$
1	1	1
2	$q^2 + q + 1$	3
3	$q^4 + q^3 + q + 1$	4
4	$q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	7
5	$q^8 + q^7 + q^6 + q^2 + q + 1$	6
6	$q^{10} + q^9 + q^8 + q^7 + q^6 + 2q^5 + q^4 + q^3 + q^2 + q + 1$	12
7	$q^{12} + q^{11} + q^{10} + q^9 + q^3 + q^2 + q + 1$	8
8	$q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	15
9	$q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^9 + q^8 + q^7 + q^4 + q^3 + q^2 + q + 1$	13
10	$q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + q^{13} + q^{12} + q^{11} + q^{10} + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	18
11	$q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^5 + q^4 + q^3 + q^2 + q + 1$	12
12	$q^{22} + q^{21} + q^{20} + q^{19} + q^{18} + q^{17} + q^{16} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + 2q^{10} + 2q^9 + q^8 + q^7 + q^6 + q^5 + q^4 + q^3 + q^2 + q + 1$	28

Theorem 1.1 has two interesting consequences. The first one concerns the polynomials $P_n(q)$. Let us state it.

COROLLARY 1.2. *For each $n \geq 1$, the polynomial $P_n(q)$ is a monic polynomial of degree $2n - 2$ with integer coefficients, and we have*

$$P_n(1) = \sigma(n) = \sum_{d|n; d \geq 1} d.$$

As is well known, the sum $\sigma(n)$ of positive divisors of n is equal to the number of subgroups of index n of the free Abelian group \mathbb{Z}^2 of rank two. Thus Theorem 1.1 and Corollary 1.2 imply that the number of ideals of codimension n of the Laurent polynomial algebra $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$, that is, of the algebra of the group \mathbb{Z}^2 , is, up to the factor $(q - 1)^2$, a q -analogue¹ of the number of subgroups of index n of \mathbb{Z}^2 .

A similar phenomenon had been observed by Bacher and the second-named author in [4]: up to a power of $q - 1$, the number of right ideals of codimension n of the algebra $\mathbb{F}_q[F_2]$ of the rank two free group F_2 is a q -analogue of the number of subgroups of index n of F_2 . In fact, it was this observation that prompted us to compute the number of ideals of codimension n of the algebra $\mathbb{F}_q[\mathbb{Z}^2]$ of the free Abelian group \mathbb{Z}^2 , that is, of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$.

In a similar context, the following holds.

- (a) By [11] (see also Section 3.1) the number of ideals of codimension n of the polynomial algebra $\mathbb{F}_q[x, y]$, which is the algebra of the free Abelian monoid \mathbb{N}^2 , is a q -analogue of the number $p(n)$ of partitions of n ; as is well known, the latter is equal to the number of ideals of the monoid \mathbb{N}^2 whose complement is of cardinality n .
- (b) In a noncommutative setting, by [24; 3] the number of right ideals of codimension n of the free algebra $\mathbb{F}_q\langle x, y \rangle$ is a q -analogue of the number of right ideals of the free monoid $\langle x, y \rangle^*$ whose complement is of cardinality n .
- (c) Similarly, by [23, Section 6.3] the number of right ideals of codimension n of the algebra $\mathbb{F}_q[F_r]$ of the free group F_r on r generators is, up to a power of $q - 1$, a q -analogue of the number of subgroups of index n of F_r .

REMARK 1.3. The commutative algebra $L_r = \mathbb{F}_q[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$ of Laurent polynomials in r variables ($r \geq 3$) provides a distinct contrast with the cases discussed. We can show that the number of right ideals of codimension 2 of L_r , which is the algebra of the free Abelian group \mathbb{Z}^r , is equal to $(q - 1)^r R_r(q)$, where

$$R_r(q) = \frac{1}{2}((q + 1)^r + (q - 1)^r) + \frac{q^r - 1}{q - 1} - 1.$$

The latter is a q -analogue of $R_r(1) = 2^{r-1} + r - 1$. Now the number of subgroups of index 2 of \mathbb{Z}^r is equal to $2^r - 1$, which is different from $R_r(1)$ when $r \geq 3$.

¹By a q -analogue of an integer r we mean a polynomial $P(q)$ in the variable q such that $P(1) = r$.

The second consequence of Theorem 1.1 expresses the generating function of the polynomials $C_n(q)$ as a nice infinite product.

COROLLARY 1.4. (a) *We have*

$$1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}.$$

(b) *The polynomials $C_n(q)$ and $P_n(q)$ are palindromic.*

The previous infinite product shows up in [12, p. 10] (see, e.g., equations (9.2) and (10.1)) and probably in other papers on basic hypergeometric series; in an algebraic geometry context it appears in [20, Thm. 4.1.3], where it is equal to the generating function of the E -polynomials of the punctual Hilbert schemes of the complex two-dimensional torus (see details in Section 6.3).

Using Corollary 1.4, we gave explicit expressions for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in the companion paper [22] (see Thms. 1.1 and 1.2 loc. cit.). We obtained a rather striking positivity result, namely the coefficients of $P_n(q)$ are all *nonnegative* integers. For completeness, we recall our formulas for the coefficients of the polynomials $C_n(q)$ and $P_n(q)$ in the Appendix.

As pointed to us by Frank Garvan, the polynomials $C_n(q)$ are related to the *crank of partitions*. Recall that the crank is a function from partitions into the integers, which explains the Ramanujan congruences modulo 11 and whose existence was conjectured by Dyson and later proved by Garvan; see [14, Section 7]. Denote as in [1] the number of partitions of n with crank m by $N_V(m, n)$. We have the following relation between the integers $N_V(m, n)$, the number $p(n)$ of all partitions of n , and the polynomials $C_n(q)$.

COROLLARY 1.5. *For each n , we have*

$$\sum_{m \in \mathbb{Z}} N_V(m, n) q^m = p(n) + \sum_{i=1}^n p(n-i) \frac{C_i(q)}{q^i}.$$

The paper is organized as follows. Section 2 is devoted to some preliminaries: we first recall the one-to-one correspondence between the ideals of the localization $S^{-1}A$ of an algebra A and certain ideals of A ; we also count tuples of polynomials subject to certain constraints over a finite field.

In Section 3 we recall Conca and Valla's parameterization of the affine cells in a decomposition of the Hilbert scheme of n points in the plane; these cells are indexed by the partitions of n . We show how to deduce a parameterization of the cells in the induced decomposition of the Hilbert scheme of n points in a Zariski open subset of the plane.

In Section 4 we apply the techniques of the preceding section to compute the number of ideals of codimension n of $\mathbb{F}_q[x, y, y^{-1}]$. In passing we give a criterion (Proposition 4.1), which will also be used in the proof of Theorem 1.1.

In Section 5 we define an invertible Gröbner cell, which is a Zariski open subset of the corresponding affine cell, and compute its cardinality over a finite field. We derive a proof of Theorem 1.1.

The proofs of Corollaries 1.2, 1.4, and 1.5 are given in Section 6.

In Appendix we briefly recall the results on the coefficients of $C_n(q)$ and $P_n(q)$ we obtained in [22].

2. Preliminaries

We fix a ground field k . By algebra we mean an associative unital k -algebra. In this paper all algebras are assumed to be commutative.

2.1. Ideals in Localizations

Let A be a (commutative) algebra, S a multiplicative submonoid of A not containing 0, and $S^{-1}A$ the corresponding localization of A . We assume that the canonical algebra map $i : A \rightarrow S^{-1}A$ is injective (this is the case, e.g., when A is a domain).

Recall the well-known correspondence between the ideals of $S^{-1}A$ and those of A (see [7, Chapter 2, Section 2, n° 4–5], [10, Prop. 2.2]).

- (a) For any ideal J of $S^{-1}A$, the set $i^{-1}(J) = J \cap A$ is an ideal of A , and we have $J = i^{-1}(J)S^{-1}A$. The map $J \mapsto i^{-1}(J)$ is an injection from the set of ideals of $S^{-1}A$ to the set of ideals of A .
- (b) An ideal I of A is of the form $i^{-1}(J)$ for some ideal J of $S^{-1}A$ if and only if for all $s \in S$, the endomorphism of A/I induced by the multiplication by s is injective.

Given an integer $n \geq 1$, an n -codimensional ideal of A is an ideal such that $\dim_k A/I = n$. For such an ideal, the previous condition (b) is then equivalent to: for all $s \in S$, the endomorphism of A/I induced by the multiplication by s is a linear isomorphism.

We leave the proof of the following lemma to the reader.

LEMMA 2.1. *If J is a finite-codimensional ideal of $S^{-1}A$, then the canonical algebra map $i : A \rightarrow S^{-1}A$ induces an algebra isomorphism*

$$A/i^{-1}(J) \cong (S^{-1}A)/J.$$

It follows that there is a bijection between the set of n -codimensional ideals of $S^{-1}A$ and the set of n -codimensional ideals I of A such that, for all $s \in S$, the endomorphism of A/I induced by the multiplication by s is a linear isomorphism. The latter assertion is equivalent to s being invertible modulo I , that is, the image of s in A/I being invertible.

The following criterion will be used in Sections 4 and 5.

LEMMA 2.2. *Let A be a commutative algebra. For any $s \in A$, let $p : A \rightarrow A/(s)$ be the natural projection onto the quotient algebra of A by the ideal generated by s . If I is an ideal of A , then s is invertible modulo I if and only if $p(I) = A/(s)$.*

Proof. If s is invertible modulo I , then there exists $t \in A$ such that $st - 1 \in I$. Hence $p(1)$ belongs to $p(I)$, which implies $p(I) = A/(s)$. Conversely, if $p(I) = A/(s)$, then $p(1) = p(u)$ for some $u \in I$. Hence $1 - u \in (s)$, which means that there is $t \in A$ such that $1 - u = st$. Thus $st \equiv 1 \pmod{I}$. \square

2.2. Counting Polynomials over a Finite Field

In this subsection we assume that $k = \mathbb{F}_q$ is a finite field of cardinality q . We will need the following in Section 5.

PROPOSITION 2.3. *Let d, h be integers ≥ 1 , and let $Q_1, \dots, Q_h \in \mathbb{F}_q[y]$ be coprime polynomials. The number of $(h + 1)$ -tuples (P, P_1, \dots, P_h) satisfying the three conditions*

- (i) P is a degree d monic polynomial with $P(0) \neq 0$,
- (ii) P_1, \dots, P_h are polynomials of degree $< d$, and
- (iii) P and $P_1Q_1 + \dots + P_hQ_h$ are coprime

is equal to

$$(q - 1)^2 q^{(h-1)d} \frac{q^{2d} - 1}{q^2 - 1}.$$

Before giving the proof, we state and prove two auxiliary lemmas.

LEMMA 2.4. *Let R be a finite commutative ring, and let $a_1, \dots, a_h \in R$ be such that $a_1R + \dots + a_hR = R$. For any $b \in R$, the number of h -tuples $(x_1, \dots, x_h) \in R^h$ such that $a_1x_1 + \dots + a_hx_h = b$ is equal to $(\text{card } R)^{h-1}$.*

Proof. The map $(x_1, \dots, x_h) \mapsto a_1x_1 + \dots + a_hx_h$ is a homomorphism $R^h \rightarrow R$ of additive groups. Since it is surjective, the number of h -tuples satisfying the assumed condition is equal to the cardinality of its kernel, which is equal to $\text{card } R^h / \text{card } R = (\text{card } R)^{h-1}$. \square

LEMMA 2.5. *Let $d \geq 1$ be an integer. The number of couples $(P, Q) \in \mathbb{F}_q[y]^2$ such that P is a degree d monic polynomial with $P(0) \neq 0$, Q is of degree $< d$, and P and Q are coprime is equal to*

$$c_d = (q - 1)^2 \frac{q^{2d} - 1}{q^2 - 1}.$$

Proof. This amounts to counting the number of couples (P, z) , where $P \in \mathbb{F}_q[y]$ is a degree d monic polynomial not divisible by y , and z is an invertible element of the quotient ring $\mathbb{F}_q[y]/(P)$.

Expanding P into a product of irreducible polynomials and using the Chinese remainder lemma, we have

$$1 + \sum_{d \geq 1} c_d t^d = \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + \sum_{k \geq 1} \text{card}(\mathbb{F}_q[y]/(P))^{\times k} t^{k \deg(P)} \right),$$

where the product is taken over all irreducible polynomials of $\mathbb{F}_q[y]$ different from y , and where $\deg(P)$ denotes the degree of P . First, observe that, for any irreducible polynomial $P \in \mathbb{F}_q[y]$, the group $(\mathbb{F}_q[y]/(P))^\times$ of invertible elements of $\mathbb{F}_q[y]/(P)$ is of cardinality $q^{k \deg(P)} - q^{(k-1) \deg(P)}$: indeed, there are $q^{k \deg(P)}$ polynomials of degree $< k \deg(P)$, and $q^{(k-1) \deg(P)}$ of them are divisible by P and hence not invertible in $\mathbb{F}_q[y]/(P)$. Consequently,

$$\begin{aligned} 1 + \sum_{d \geq 1} c_d t^d &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + (1 - q^{-\deg(P)}) \sum_{k \geq 1} (qt)^{k \deg(P)} \right) \\ &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \left(1 + (1 - q^{-\deg(P)}) \frac{(qt)^{\deg(P)}}{1 - (qt)^{\deg(P)}} \right) \\ &= \prod_{\substack{P \text{ irreducible} \\ P \neq y}} \frac{1 - t^{\deg(P)}}{1 - (qt)^{\deg(P)}}. \end{aligned}$$

On one hand, the infinite product $\prod_{\substack{P \text{ irreducible} \\ P \neq y}} (1 - t^{\deg(P)})^{-1}$ is equal to the zeta function $Z_{\mathbb{A}^1 \setminus \{0\}}(t)$ of the affine line minus a point. On the other hand,

$$Z_{\mathbb{A}^1 \setminus \{0\}}(t) = \frac{Z_{\mathbb{A}^1}(t)}{Z_{\{0\}}(t)} = \frac{1 - t}{1 - qt}.$$

Therefore

$$1 + \sum_{d \geq 1} c_d t^d = \frac{1 - qt}{1 - q^2 t} \bigg/ \frac{1 - t}{1 - qt} = \frac{(1 - qt)^2}{(1 - t)(1 - q^2 t)}.$$

Subtracting 1 from both sides, we obtain

$$\sum_{d \geq 1} c_d t^d = (q - 1)^2 \frac{t}{(1 - t)(1 - q^2 t)},$$

from which it is easy to derive the desired formula for c_d . □

Proof of Proposition 2.3. We have to count the number of the $(h + 2)$ -tuples (P, Q, P_1, \dots, P_h) such that P is a degree d monic polynomial with $P(0) \neq 0$, Q is a polynomial of degree $< d$ and coprime to P , each polynomial P_i is of degree $< d$, and $\sum_{i=1}^h P_i Q_i \equiv Q$ modulo P .

By Lemma 2.5 the number of couples (P, Q) satisfying these conditions is equal to $(q - 1)^2 (q^{2d} - 1) / (q^2 - 1)$. Since $\text{card } \mathbb{F}_q[y]/(P) = q^d$, by Lemma 2.4 we have $q^{d(h-1)}$ choices for the h -tuples (P_1, \dots, P_h) . The number we wish to count is the product of the two previous ones. □

3. The Hilbert Scheme of Points in a Zariski Open Subset of the Plane

Let k be a field. As is well known, the ideals of codimension n of an affine k -algebra A are in bijection with the k -points of the Hilbert scheme parameterizing

finite subschemes of colength n of the spectrum of A . For instance, the ideals of codimension n of the polynomial algebra $k[x, y]$ are in bijection with the k -points of the Hilbert scheme $\text{Hilb}^n(\mathbb{A}_k^2)$ of n points on the affine plane. Similarly, the ideals of codimension n of the Laurent polynomial algebra $k[x, y, x^{-1}, y^{-1}]$ are in bijection with the k -points of the Hilbert scheme $\text{Hilb}^n((\mathbb{A}_k^1 \setminus \{0\}) \times (\mathbb{A}_k^1 \setminus \{0\}))$ of n points on the two-dimensional torus, which is a Zariski open subset of the plane.

In this paragraph we prove that the Hilbert scheme of n points in a Zariski open subset of the plane is an open subscheme of the Hilbert scheme of n points in the plane and show how to determine it explicitly.

3.1. Parameterizing the Finite-Codimensional Ideals of $k[x, y]$

Computing the homology of the Hilbert scheme $\text{Hilb}^n(\mathbb{A}_k^2)$, Ellingsrud and Strømme [11] showed that it has a cellular decomposition indexed by the partitions λ of n , each cell C_λ being an affine space of dimension $n + \ell(\lambda)$, where $\ell(\lambda)$ is the length of λ . Earlier results by Białyński-Birula [5; 6] on smooth varieties with k^\times -actions imply the same decomposition; the cells C_λ are sometimes called “Białyński-Birula cells”.

It follows that, in the particular case where $k = \mathbb{F}_q$ is a finite field of cardinality q , the number $A_n(q)$ of ideals of $\mathbb{F}_q[x, y]$ of codimension n is finite and given by the polynomial

$$A_n(q) = \sum_{\lambda \vdash n} q^{n+\ell(\lambda)}, \tag{3.1}$$

where the sum runs over all partitions λ of n (we indicate this by the notation $\lambda \vdash n$ or by $|\lambda| = n$). The polynomial $A_n(q)$ clearly has nonnegative integer coefficients, its degree is $2n$, and $A_n(1) = p(n)$ is equal to the number of partitions of n (for more on the polynomials $A_n(q)$, see Remark 4.7).

For our purposes, we need an explicit description of the affine cells C_λ . Let us recall a parameterization by Conca and Valla [8].

Given a positive integer n , there is a well-known bijection between the partitions of n and the monomial ideals of codimension n of $k[x, y]$. The correspondence is as follows: with a partition λ of n , we associate the sequence

$$0 = m_0 < m_1 \leq \dots \leq m_t$$

of integers counting from right to left the boxes in each column of the Ferrers diagram of λ ; we have $m_1 + \dots + m_t = n$. Then the associated monomial ideal I_λ^0 is given by

$$I_\lambda^0 = (x^t, x^{t-1}y^{m_1}, \dots, x y^{m_{t-1}}, y^{m_t}). \tag{3.2}$$

(Note that the generating set in the right-hand side of (3.2) is in general not minimal.) The set $\mathcal{B}_\lambda = \{x^i y^j \mid 0 \leq i < t, 0 \leq j < m_i\}$ induces a linear basis of the n -dimensional quotient algebra $k[x, y]/I_\lambda^0$.

Consider the lexicographic ordering on the monomials $x^i y^j$ given by

$$1 < y < y^2 < \dots < x < xy < xy^2 < \dots < x^2 < x^2y < x^2y^2 < \dots$$

Then the cell C_λ , called *Gröbner cell* in [8], is by definition the set of ideals I of $k[x, y]$ such that the dominating terms (for this ordering) of the elements of I generate the monomial ideal I_λ^0 . It was proved in [11] that C_λ is an affine space.

Here is how Conca and Valla explicitly parameterize C_λ . Given a partition λ of n and the associated sequence $0 = m_0 < m_1 \leq \dots \leq m_t$, they first define the sequence of integers d_1, \dots, d_t by

$$d_i = m_i - m_{i-1} \geq 0. \tag{3.3}$$

We have $d_1 = m_1 > 0$.

Later we shall also need the integer

$$v(\lambda) = \text{card}\{i = 1, \dots, t \mid d_i \geq 1\}; \tag{3.4}$$

it is equal to the number of distinct values of the sequence $m_1 \leq \dots \leq m_t$. Note that $v(\lambda) \geq 1$; moreover, $v(\lambda) = 1$ if and only if the partition is “rectangular”, that is, $m_1 = \dots = m_t (> 0)$.

Let T_λ be the set of $(t + 1) \times t$ -matrices $(p_{i,j})$ with entries in the one-variable polynomial algebra $k[y]$ satisfying the following conditions: $p_{i,j} = 0$ if $i < j$, the degree of $p_{i,j}$ is less than d_j if $i \geq j$ and $d_j \geq 1$, and $p_{i,j} = 0$ for all i if $d_j = 0$. The set T_λ is an affine space of dimension $n + \ell(\lambda)$.

Now consider the $(t + 1) \times t$ -matrix

$$M_\lambda = \begin{pmatrix} y^{d_1} + p_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ p_{2,1} - x & y^{d_2} + p_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ p_{3,1} & p_{3,2} - x & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{i-1,1} & p_{i-1,2} & \dots & y^{d_{i-1}} + p_{i-1} & 0 & 0 & \dots & 0 \\ p_{i,1} & p_{i,2} & \dots & p_{i,i-1} - x & y^{d_i} + p_i & 0 & \dots & 0 \\ p_{i+1,1} & p_{i+1,2} & \dots & p_{i+1,i-1} & p_{i+1,i} - x & y^{d_{i+1}} + p_{i+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{t,1} & p_{t,2} & \dots & p_{t,i-1} & p_{t,i} & p_{t,i+1} & \dots & y^{d_t} + p_t \\ p_{t+1,1} & p_{t+1,2} & \dots & p_{t+1,i-1} & p_{t+1,i} & p_{t+1,i+1} & \dots & p_{t+1,t} - x \end{pmatrix}, \tag{3.5}$$

where for simplicity we set $p_i = p_{i,i}$.

By [8, Thm. 3.3] the map sending the polynomial matrix $(p_{i,j}) \in T_\lambda$ to the ideal I_λ of $k[x, y]$ generated by all t -minors (the maximal minors) of the matrix M_λ is a bijection of T_λ onto C_λ . These minors are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$ and in the variables x, y .

3.2. Localizing

Let S be a multiplicative submonoid of $k[x, y]$ not containing 0. We assume that S has a finite generating set Σ . We further concentrate on two cases, $\Sigma = \{y\}$ (in Section 4) and $\Sigma = \{x, y\}$ (in Section 5).

It follows from Section 2 that the set of n -codimensional ideals of the localization $S^{-1}k[x, y]$ can be identified with the subset of $\text{Hilb}^n(\mathbb{A}_k^2)$ consisting of the n -codimensional ideals I of $k[x, y]$ such that, for all $s \in S$, the endomorphism μ_s of $k[x, y]/I$ induced by the multiplication by s is a linear isomorphism. The latter is equivalent to $\det \mu_s \neq 0$ for all $s \in \Sigma$.

By the considerations of Section 3.1, the set of n -codimensional ideals of the algebra $S^{-1}k[x, y]$ is the disjoint union

$$\coprod_{\lambda \vdash n} C_\lambda^\Sigma,$$

where C_λ^Σ is the Zariski open subset of the affine Gröbner cell C_λ consisting of the points satisfying $\det \mu_s \neq 0$ for all $s \in \Sigma$.

Consequently, the Hilbert scheme $\text{Hilb}^n(\text{Spec}(S^{-1}k[x, y]))$ parameterizing subschemes of colength n in $\text{Spec}(S^{-1}k[x, y])$ is an open subscheme of $\text{Hilb}^n(\mathbb{A}_k^2)$ and hence an open subscheme of $\text{Hilb}^n(\mathbb{P}_k^2)$. Since by [13; 16] the latter is smooth and projective, $\text{Hilb}^n(\text{Spec}(S^{-1}k[x, y]))$ is a smooth quasi-projective variety.

The endomorphism μ_x (resp. μ_y) of $k[x, y]/I$ induced by the multiplication by x (resp. by y) can be expressed as a matrix in the basis \mathcal{B}_λ . Observe that the entries of such a matrix are polynomial expressions with integer coefficients in the coefficients of the $p_{i,j}$. Therefore, if any $s \in \Sigma$ is a linear combination with integer coefficients of monomials in the variables x, y , then the Hilbert scheme $\text{Hilb}^n(\text{Spec}(S^{-1}k[x, y]))$ is defined over \mathbb{Z} as a variety.

In particular, the schemes $\text{Hilb}^n(\mathbb{A}_k^1 \times (\mathbb{A}_k^1 \setminus \{0\}))$ and $\text{Hilb}^n((\mathbb{A}_k^1 \setminus \{0\})^2)$ are smooth quasi-projective varieties defined over \mathbb{Z} .

EXAMPLE 3.1. Let λ be the unique self-conjugate partition of 3. In this case, $t = 2, m_1 = 1, m_2 = 2$, and hence $d_1 = d_2 = 1$. The corresponding matrix M_λ , as in (3.5), is

$$M_\lambda = \begin{pmatrix} y + a & 0 \\ b - x & y + d \\ c & e - x \end{pmatrix},$$

where a, b, c, d, e are scalars. The associated Gröbner cell C_λ is a five-dimensional affine space parameterized by these five scalars. The ideal I_λ is generated by the maximal minors of the matrix, namely by $(b - x)(e - x) - c(y + d)$, $(e - x)(y + a)$, and $(y + a)(y + d)$. It follows that modulo I_λ we have the relations

$$\begin{aligned} x^2 &\equiv (b + e)x + cy + (cd - be), \\ xy &\equiv -ax + ey + ae, \\ y^2 &\equiv -(a + d)y - ad. \end{aligned}$$

In the basis $\mathcal{B}_\lambda = \{x, y, 1\}$ the multiplication endomorphisms μ_x and μ_y can be expressed as the matrices

$$\mu_x = \begin{pmatrix} b + e & -a & 1 \\ c & e & 0 \\ cd - be & ae & 0 \end{pmatrix} \quad \text{and} \quad \mu_y = \begin{pmatrix} -a & 0 & 0 \\ e & -(a + d) & 1 \\ ae & -ad & 0 \end{pmatrix}.$$

We have $\det \mu_x = e(ac - cd + be)$ and $\det \mu_y = -ad^2$.

It follows from these computations that if, for instance, $\Sigma = \{x, y\}$, then C_λ^Σ is the complement in the affine space \mathbb{A}_k^5 of the union of the three hyperplanes $a = 0, d = 0$, and $e = 0$ and of the quadric hypersurface $ac - cd + be = 0$.

4. The Punctual Hilbert Scheme of the Complement of a Line in an Affine Plane

In this section we apply the considerations of the previous section to the case $\Sigma = \{y\}$. Here S is the multiplicative submonoid of $k[x, y]$ generated by y and $S^{-1}k[x, y] = k[x, y, y^{-1}] = k[x][y, y^{-1}]$.

By Section 3.2, the Hilbert scheme $\text{Hilb}^n(\mathbb{A}_k^1 \times (\mathbb{A}_k^1 \setminus \{0\}))$, that is, the set of n -codimensional ideals of $k[x, y, y^{-1}]$, is the disjoint union over the partitions λ of n of the sets C_λ^y , where C_λ^y consists of the ideals $I \in C_\lambda$ such that y is invertible in $k[x, y]/I$. We call C_λ^y the *semi-invertible Gröbner cell* associated with the partition λ .

4.1. A Criterion for the Invertibility of y

Let $p_y : k[x, y] \rightarrow k[x]$ be the algebra map sending x to itself and y to 0. Then by Lemma 2.2, the set C_λ^y consists of the ideals $I \in C_\lambda$ such that $p_y(I) = k[x]$.

Recall from Section 3.1 that I_λ is generated by the maximal minors of the matrix M_λ of (3.5), namely by the polynomials $f_0(x, y), \dots, f_t(x, y)$, where we define $f_i(x, y)$ to be the determinant of the $t \times t$ -matrix obtained from M_λ by deleting its $(i + 1)$ th row. Then the ideal $p_y(I_\lambda)$ can be identified with the ideal of $k[x]$ generated by the polynomials $f_0(x, 0), \dots, f_t(x, 0) \in k[x]$ obtained by setting $y = 0$. We need to determine under what conditions this ideal is equal to the whole algebra $k[x]$.

Recall the entries of the matrix M_λ and particularly the polynomials $p_{i,j}$ and $p_i = p_{i,i} \in k[y]$. Let $a_{i,j} = p_{i,j}(0)$ be the constant term of $p_{i,j}$. As before, we set $a_i = a_{i,i} = p_i(0)$. Note that $a_j = 1$ and $a_{i,j} = 0$ for all $i \neq j$ whenever $d_j = 0$.

Then $f_0(x, 0), \dots, f_t(x, 0)$ are the maximal minors of the matrix

$$M_\lambda^y = \begin{pmatrix} a_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_{2,1-x} & a_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ a_{3,1} & a_{3,2-x} & a_3 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & a_{i-1,2} & a_{i-1,3} & \dots & a_{i-1} & 0 & 0 & \dots & 0 \\ a_{i,1} & a_{i,2} & a_{i,3} & \dots & a_{i,i-1-x} & a_i & 0 & \dots & 0 \\ a_{i+1,1} & a_{i+1,2} & a_{i+1,3} & \dots & a_{i+1,i-1} & a_{i+1,i-x} & a_{i+1} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{r,1} & a_{r,2} & a_{r,3} & \dots & a_{r,i-1} & a_{r,i} & a_{r,i+1} & \dots & a_r \\ a_{t+1,1} & a_{t+1,2} & a_{t+1,3} & \dots & a_{t+1,i-1} & a_{t+1,i} & a_{t+1,i+1} & \dots & a_{t+1,t-x} \end{pmatrix}.$$

To be precise, $f_i(x, 0)$ is the determinant of the square matrix obtained from M_λ^y by deleting its $(i + 1)$ th row.

The criterion we need is the following.

PROPOSITION 4.1. *We have $p_y(I_\lambda) = k[x]$ if and only if $a_i \neq 0$ for all $i = 1, \dots, t$ such that $d_i \geq 1$ (equivalently, for all $i = 1, \dots, t$).*

Proof. Since $a_i = 1$ when $d_i = 0$, it is equivalent to prove that $p_y(I_\lambda) = k[x]$ if and only if $a_1 a_2 \cdots a_t \neq 0$.

Set $I_x = p_y(I_\lambda) \subset k[x]$. The condition $a_1 a_2 \cdots a_t \neq 0$ is sufficient. Indeed, the last polynomial $f_t(x, 0)$ is the determinant of a lower triangular matrix whose diagonal entries are the scalars a_i ; hence, $f_t(x, 0) = a_1 a_2 \cdots a_t$. Thus, if $f_t(x, 0)$ is nonzero, then $I_x = k[x]$.

To check the necessity of the condition, we will prove that, for each $i = 1, \dots, t$, the vanishing of the scalar a_i implies that the ideal I_x is contained in a proper ideal generated by a minor of M_λ^y .

If $a_1 = 0$, then $f_1(x, 0) = \cdots = f_t(x, 0) = 0$ since these are determinants of matrices whose first row is zero. It follows that I_x is the principal ideal generated by the characteristic polynomial $f_0(x, 0)$, which is of degree $t \geq 1$. Hence, I_x is a proper ideal of $k[x]$.

Let now $i \geq 2$. If for $k \geq i$, we delete the $(k + 1)$ th row of M_λ^y , then we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1 & 0 \\ * & M_2^{(k)} \end{pmatrix},$$

where M_1 is the square submatrix of M_λ^y corresponding to the rows $1, \dots, i$ and to the columns $1, \dots, i$; this is a lower triangular matrix whose diagonal entries are a_1, \dots, a_i . Consequently, if $a_i = 0$, then $f_k(x, 0) = 0$ for all $k \geq i$.

Under the same condition $a_i = 0$, if we delete the $(k + 1)$ th row of M_λ^y for $k < i$, then we obtain a lower block-triangular matrix of the form

$$\begin{pmatrix} M_1^{(k)} & 0 \\ * & M_2 \end{pmatrix},$$

where M_2 is the square submatrix of M_λ^y corresponding to the rows $i + 1, \dots, t + 1$ and to the columns i, \dots, t :

$$M_2 = \begin{pmatrix} a_{i+1,i} - x & a_{i+1} & \cdots & 0 & 0 \\ a_{i+2,i} & a_{i+2,i+1} - x & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{t,i} & \cdots & \cdots & a_{t,t-1} - x & a_t \\ a_{t+1,i} & a_{t+1,i+1} & \cdots & a_{t+1,t-1} & a_{t+1,t} - x \end{pmatrix}.$$

Consequently, the polynomials $f_k(x, 0)$ for $k < i$ are all divisible by the determinant of M_2 . Thus, I_x is contained in the ideal generated by $\det(M_2)$, which is a characteristic polynomial of degree $t - i + 1$. Since $t - i + 1 \geq 1$ for all $i = 1, \dots, t$, we have $I_x \neq k[x]$. □

As an immediate consequence of Section 3.2 and of Proposition 4.1, we obtain the following:

COROLLARY 4.2. *The set of n -codimensional ideals of $k[x, y, y^{-1}]$ is the disjoint union*

$$\coprod_{\lambda \vdash n} C_\lambda^y,$$

where C_λ^y is the complement in the affine Gröbner cell C_λ of the union of the hyperplanes $a_i = 0$ where i runs over all integers $i = 1, \dots, t$ such that $d_i \geq 1$.

4.2. On the Number of Finite-Codimensional Ideals of $\mathbb{F}_q[x, y, y^{-1}]$

Recall that the positive integer $v(\lambda)$ defined by (3.4) is the number of distinct columns of the partition λ .

PROPOSITION 4.3. Let $k = \mathbb{F}_q$. For each partition λ of n , the set C_λ^y is finite, and its cardinality is given by

$$\text{card } C_\lambda^y = (q - 1)^{v(\lambda)} q^{n + \ell(\lambda) - v(\lambda)}.$$

Proof. By Corollary 4.2 the set C_λ^y is parameterized by $n + \ell(\lambda)$ parameters subject to the sole condition that $v(\lambda)$ of them are not zero. □

COROLLARY 4.4. For each integer $n \geq 1$, the number $B_n(q)$ of n -codimensional ideals of $\mathbb{F}_q[x, y, y^{-1}]$ is equal to $(q - 1)q^n B_n^\circ(q)$, where

$$B_n^\circ(q) = \sum_{\lambda \vdash n} (q - 1)^{v(\lambda) - 1} q^{\ell(\lambda) - v(\lambda)}.$$

Note that $B_n^\circ(q)$ is a polynomial in q since $v(\lambda) \geq 1$ and $\ell(\lambda) \geq v(\lambda)$ for all partitions. It is of degree $n - 1$ and has integer coefficients. The coefficients of $B_n^\circ(q)$ may be negative, as we can see in Table 3.

REMARK 4.5. Let v_n be the valuation of the polynomial $B_n^\circ(q)$, that is, the maximal integer r such that q^r divides $B_n^\circ(q)$. We conjecture that $v_n = 0, 1, \text{ or } 2$ and

Table 3 The polynomials $B_n^\circ(q)$

n	$B_n^\circ(q)$	$B_n^\circ(1)$	$B_n^\circ(-1)$
1	1	1	1
2	$q + 1$	2	0
3	$q^2 + q$	2	0
4	$q^3 + q^2 + q$	3	-1
5	$q^4 + q^3 + q^2 - 1$	2	0
6	$q^5 + q^4 + q^3 + q^2$	4	0
7	$q^6 + q^5 + q^4 + q^3 - q - 1$	2	0
8	$q^7 + q^6 + q^5 + q^4 + q^3 - q$	4	0
9	$q^8 + q^7 + q^6 + q^5 + q^4 - q^2 - q$	3	1
10	$q^9 + q^8 + q^7 + q^6 + q^5 + q^4 - q^2 - q$	4	0
11	$q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 - q^3 - 2q^2 - q$	2	0
12	$q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 + q^5 - q^3 - q^2 + 1$	6	0

that the infinite word $v_1v_2v_3\dots$ is equal to $0\prod_{n=1}^\infty 01^{2n}02^n$. This conjecture is supported by computer calculations.

Let us now give a product formula for the generating function of the sequence of polynomials $B_n(q)$ and an arithmetical interpretation for two values of $B_n(q)$.

THEOREM 4.6. (a) *Let $B_n(q)$ be the number of ideals of $\mathbb{F}_q[x, y, y^{-1}]$ of codimension n . We have*

$$1 + \sum_{n \geq 1} \frac{B_n(q)}{q^n} t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 - qt^i}.$$

(b) *Let $B_n^\circ(q)$ be the polynomial $B_n^\circ(q) = (q - 1)^{-1}q^{-n}B_n(q)$. It has integer coefficients and satisfies*

$$B_n^\circ(1) = \sigma_0(n),$$

where $\sigma_0(n)$ is the number of divisors of n , and

$$B_n^\circ(-1) = \begin{cases} (-1)^{k-1} & \text{if } n = k^2 \text{ for some integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (a) Since an analogous proof will be used in Remark 4.7 and Section 6.2, we give here a detailed proof. Let X be a set, and let M be the free Abelian monoid on X (X is called a basis of M). We say that a function $\varphi : M \rightarrow R$ from M to a ring R is *multiplicative* if $\varphi(uv) = \varphi(u)\varphi(v)$ for all couples $(u, v) \in M^2$ of words having no common basis element. Under this condition, it is easy to check the following identity:

$$\sum_{w \in M} \varphi(w) = \prod_{x \in X} \left(1 + \sum_{e \geq 1} \varphi(x^e) \right). \tag{4.1}$$

Now, identifying each partition with its planar diagram, we consider a partition λ as a union of rectangular partitions i^{e_i} , with e_i parts of length i , for $e_i \geq 1$ and distinct $i \geq 1$, which we denote by the formal product $\lambda = \prod_{i \geq 1} i^{e_i}$. Thus the set of partitions is equal to the free Abelian monoid on $X = \mathbb{N} \setminus \{0\}$ (viewed as a set). Before we apply (4.1), let us remark that $|\lambda| = \sum_i i e_i$ and $\ell(\lambda) = \sum_i e_i$. Moreover, the multisets $\{e_i \mid i \geq 1\}$ and $\{d_i \mid i \geq 1\}$ are equal (recall that the integers d_i are those associated with λ in (3.3)); therefore, $v(\lambda) = \sum_{i, d_i \geq 1} 1 = \text{card}\{i \mid e_i \geq 1\}$.

Let s be a new variable. The function $\lambda \mapsto \text{card } C_\lambda^y s^{|\lambda|}$ computed in Proposition 4.3 is clearly multiplicative. Applying (4.1), we obtain

$$\begin{aligned} 1 + \sum_{n \geq 1} B_n(q) s^n &= 1 + \sum_{|\lambda| \geq 1} \text{card } C_\lambda^y s^{|\lambda|} \\ &= \prod_{i \geq 1} \left(1 + \sum_{e \geq 1} \text{card } C_{i^e}^y s^{ie} \right) \end{aligned}$$

$$\begin{aligned}
 &= \prod_{i \geq 1} \left(1 + \sum_{e \geq 1} (q-1)q^{ie+e-1}s^{ie} \right) \\
 &= \prod_{i \geq 1} \left(1 + (q-1)q^{-1} \sum_{e \geq 1} (q^{i+1}s^i)^e \right) \\
 &= \prod_{i \geq 1} \left(1 + (q-1)q^{-1} \frac{q^{i+1}s^i}{1 - q^{i+1}s^i} \right) \\
 &= \prod_{i \geq 1} \frac{(1 - q^{i+1}s^i) + (q-1)q^i s^i}{1 - q^{i+1}s^i} \\
 &= \prod_{i \geq 1} \frac{1 - q^i s^i}{1 - q^{i+1}s^i}.
 \end{aligned}$$

Finally, replace s by $q^{-1}t$.

(b) To compute $B_n^\circ(1)$, we use the formula of Corollary 4.4. Since the value at $q = 1$ of $(q-1)^{v(\lambda)-1}$ is 1 if $v(\lambda) = 1$ and 0 otherwise and since $v(\lambda) = 1$ if and only if $m_1 = \dots = m_t = d$, in which case $dt = n$, we have

$$B_n^\circ(1) = \sum_{dt=n} 1 = \sum_{d|n, d \geq 1} 1 = \sigma_0(n).$$

For $B_n^\circ(-1)$, we use the infinite product expansion of Item (a): replacing $B_n(q)$ by $(q-1)q^n B_n^\circ(q)$, we obtain

$$1 + \sum_{n \geq 1} (q-1)B_n^\circ(q)t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 - qt^i}.$$

Setting $q = -1$ yields

$$1 - 2 \sum_{n \geq 1} B_n^\circ(-1)t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 + t^i}.$$

Now recall the following identity of Gauss (see [12, (7.324)] or [17, 19.9 (i)]):

$$\prod_{i \geq 1} \frac{1 - t^i}{1 + t^i} = \sum_{k \in \mathbb{Z}} (-1)^k t^{k^2}. \tag{4.2}$$

It follows that

$$1 - 2 \sum_{n \geq 1} B_n^\circ(-1)t^n = 1 + 2 \sum_{k \geq 1} (-1)^k t^{k^2},$$

which allows us to conclude. □

REMARK 4.7. The results of Theorem 4.6 should be compared to the following ones concerning the number $A_n(q)$ of ideals of $\mathbb{F}_q[x, y]$ of codimension n . Proceeding as in the proof of Theorem 4.6, we deduce from (3.1) that

$$1 + \sum_{n \geq 1} A_n(q)s^n = \prod_{i \geq 1} \frac{1}{1 - q^{i+1}s^i}.$$

Setting $q = -1$, we have

$$1 + \sum_{n \geq 1} A_n(-1)s^n = \prod_{i \geq 1} \frac{1}{1 - (-1)^{i+1}s^i} = \prod_{m \geq 1} \frac{1}{(1 - s^{2m-1})(1 + s^{2m})}. \quad (4.3)$$

Multiplying by $\prod_{m \geq 1} (1 + s^{2m})^{-1}$ both sides of the Euler identity

$$\prod_{m \geq 1} \frac{1}{1 - s^{2m-1}} = \prod_{i \geq 1} (1 + s^i)$$

(see [17, (19.4.7)]), we deduce that the right-hand side of (4.3) is equal to the infinite product

$$\prod_{m \geq 1} (1 + s^{2m-1}).$$

Thus by [2, Table 14.1, p. 310] or [17, (19.4.4)], the value $A_n(-1)$ is equal to the number² of partitions of n with unequal odd parts. Note that $A_n(1)$ is equal to the number³ of partitions of n . See Table 4 for a list of the polynomials $A_n(q)$ ($1 \leq n \leq 12$).

Table 4 The polynomials $A_n(q)$

n	$A_n(q)$	$A_n(1)$	$A_n(-1)$
1	q^2	1	1
2	$q^4 + q^3$	2	0
3	$q^6 + q^5 + q^4$	3	1
4	$q^8 + q^7 + 2q^6 + q^5$	5	1
5	$q^{10} + q^9 + 2q^8 + 2q^7 + q^6$	7	1
6	$q^{12} + q^{11} + 2q^{10} + 3q^9 + 3q^8 + q^7$	11	1
7	$q^{14} + q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 3q^9 + q^8$	15	1
8	$q^{16} + q^{15} + 2q^{14} + 3q^{13} + 5q^{12} + 5q^{11} + 4q^{10} + q^9$	22	2
9	$q^{18} + q^{17} + 2q^{16} + 3q^{15} + 5q^{14} + 6q^{13} + 7q^{12} + 4q^{11} + q^{10}$	30	2
10	$q^{20} + q^{19} + 2q^{18} + 3q^{17} + 5q^{16} + 7q^{15} + 9q^{14} + 8q^{13} + 5q^{12} + q^{11}$	42	2
11	$q^{22} + q^{21} + 2q^{20} + 3q^{19} + 5q^{18} + 7q^{17} + 10q^{16} + 11q^{15} + 10q^{14} + 5q^{13} + q^{12}$	56	2
12	$q^{24} + q^{23} + 2q^{22} + 3q^{21} + 5q^{20} + 7q^{19} + 11q^{18} + 13q^{17} + 15q^{16} + 12q^{15} + 6q^{14} + q^{13}$	77	3

²See Sequence A000700 in [25].

³See Sequence A000041 in [25].

5. Invertible Gröbner Cells

Let $\text{Hilb}^n((\mathbb{A}_k^1 \setminus \{0\})^2)$ be the Hilbert scheme parameterizing finite subschemes of colength n of the two-dimensional torus, that is, of the complement of two distinct intersecting lines in the affine plane. Its k -points are in bijection with the set of ideals of $k[x, y, x^{-1}, y^{-1}]$ of codimension n . By Section 3.2 this set of ideals is the disjoint union over the partitions λ of n of the sets $C_\lambda^{x,y}$, where $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that both x and y are invertible in $k[x, y]/I$. We call $C_\lambda^{x,y}$ the *invertible Gröbner cell* associated with the partition λ .

When the ground field is finite, so is $C_\lambda^{x,y}$. The aim of this section is to compute the cardinality of $C_\lambda^{x,y}$ when $k = \mathbb{F}_q$.

5.1. The Cardinality of an Invertible Gröbner Cell

Recall the nonnegative integers d_1, \dots, d_t defined by (3.3) and the positive integer $v(\lambda)$ defined by (3.4). We now give a formula for $\text{card } C_\lambda^{x,y}$.

THEOREM 5.1. *Let $k = \mathbb{F}_q$, let n be an integer ≥ 1 , and let λ be a partition of n . Then*

$$\text{card } C_\lambda^{x,y} = (q - 1)^{2v(\lambda)} q^{n-\ell(\lambda)} \prod_{\substack{i=1,\dots,t \\ d_i \geq 1}} \frac{q^{2d_i} - 1}{q^2 - 1}.$$

The theorem will be proved in Section 5.3. It has the following straightforward consequences.

COROLLARY 5.2. *Let $k = \mathbb{F}_q$, and let λ be a partition of n .*

- (a) $\text{card } C_\lambda^{x,y}$ is a monic polynomial in q with integer coefficients; it is of degree $n + \ell(\lambda)$.
- (b) The polynomial $\text{card } C_\lambda^{x,y}$ is divisible by $(q - 1)^2$. The quotient

$$P_\lambda(q) = \frac{\text{card } C_\lambda^{x,y}}{(q - 1)^2}$$

is a monic polynomial in q with integer coefficients and of degree $n + \ell(\lambda) - 2$.

- (c) If the partition λ is rectangular, that is, if $v(\lambda) = 1$, in which case $d_2 = \dots = d_t = 0$ and $d = d_1$ is a divisor of n , then

$$P_\lambda(q) = q^{n-d} \frac{q^{2d} - 1}{q^2 - 1} = q^{n-d} (1 + q^2 + \dots + q^{2d-2}).$$

In this case, $P_\lambda(1) = d$.

- (d) If $v(\lambda) \geq 2$, then $P_\lambda(q)$ is divisible by $(q - 1)^2$, and $P_\lambda(1) = 0$.

REMARK 5.3. The polynomials $P_\lambda(q)$ may have negative coefficients. For instance, if λ is the partition of 4 corresponding to $t = 2, d_1 = 1, d_2 = 2$, then

$$P_\lambda(q) = q^5 - 2q^4 + 2q^3 - 2q^2 + q.$$

The rest of the section is devoted to the proof of Theorem 5.1.

5.2. A Criterion for the Invertibility of x

In Section 4 we introduced the algebra map $p_y : k[x, y] \rightarrow k[x]$ sending x to itself and y to 0. Similarly, let $p_x : k[x, y] \rightarrow k[x]$ be the algebra map sending x to 0 and y to itself. Then by Lemma 2.2, the set $C_\lambda^{x,y}$ consists of the ideals $I \in C_\lambda$ such that $p_x(I) = k[y]$ and $p_y(I) = k[x]$. We already have a criterion for $p_y(I) = k[x]$ (see Proposition 4.1). We shall now give a necessary and sufficient condition for $p_x(I)$ to be equal to $k[y]$.

Resuming the notation of Section 4, we see that $p_x(I)$ can be identified with the ideal of $k[y]$ generated by the polynomials $f_0(0, y), \dots, f_t(0, y) \in k[y]$ obtained from the polynomials $f_0(x, y), \dots, f_t(x, y)$ by setting $x = 0$. The polynomials $f_0(0, y), \dots, f_t(0, y)$ are the maximal minors of the matrix M_λ^x obtained from the matrix M_λ of (3.5) by setting $x = 0$.

Let μ_i be the determinant of the submatrix M_i of M_λ^x corresponding to the rows $(i + 1), \dots, (t + 1)$ and to the columns i, \dots, t . We have $\mu_t = p_{t+1,t}$ and

$$\mu_i = \begin{vmatrix} p_{i+1,i} & y^{d_{i+1}} + p_{i+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{t,i} & p_{t,i+1} & \cdots & y^{d_t} + p_t \\ p_{t+1,i} & p_{t+1,i+1} & \cdots & p_{t+1,t} \end{vmatrix}$$

if $1 \leq i < t$. Expanding μ_i along its first column, we obtain

$$\mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i} q_{i+j,i}, \tag{5.1}$$

where

$$q_{i+j,i} = \begin{cases} \mu_{i+1} & \text{if } j = 1, \\ (-1)^{j-1} (y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_{i+j-1}} + p_{i+j-1}) \mu_{i+j} & \text{if } 1 < j < t - i + 1, \\ (-1)^{t-i} (y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_{t-1}} + p_{t-1}) (y^{d_t} + p_t) & \text{if } j = t - i + 1. \end{cases} \tag{5.2}$$

Observe also that

$$f_i(0, y) = \begin{cases} \mu_1 & \text{if } i = 0, \\ (y^{d_1} + p_1) \cdots (y^{d_i} + p_i) \mu_{i+1} & \text{if } 1 \leq i < t, \\ (y^{d_1} + p_1) \cdots (y^{d_t} + p_t) & \text{if } i = t. \end{cases} \tag{5.3}$$

LEMMA 5.4. *If $1 \leq i \leq j \leq t$, then μ_i belongs to the ideal $(\mu_j, y^{d_j} + p_j)$ generated by μ_j and $(y^{d_j} + p_j)$.*

Proof. The case $i = j$ is obvious. Otherwise, consider the matrix M_i whose determinant is μ_i ; the column of M_i containing the entry $y^{d_j} + p_j$ can be written as the sum of a column containing only the entry $y^{d_j} + p_j$, the other entries being zero, and of a column whose top entry is zero and the bottom ones form the first

column of the matrix M_j whose determinant is μ_j . Therefore by the multilinearity property of determinants, μ_i is the sum of a determinant that is a multiple of $y^{d_j} + p_j$ and of another determinant that is a multiple of μ_j ; indeed, this second determinant is block-triangular with one diagonal block equal to μ_j . \square

Here is our criterion for the invertibility of x .

PROPOSITION 5.5. *We have $p_x(I_\lambda) = k[y]$ if and only if $y^{d_i} + p_i$ and μ_i are coprime for all $i = 1, \dots, t$.*

Proof. (a) Let us first check the sufficiency. The fact that $y^{d_i} + p_i$ and μ_i are coprime implies that by (5.3) the gcd of $f_t(0, y)$ and of $f_{t-1}(0, y)$ is $(y^{d_t} + p_t) \cdots (y^{d_{t-1}} + p_{t-1})$. Now the gcd of the latter and of $f_{t-2}(0, y)$ is $(y^{d_t} + p_t) \cdots (y^{d_{t-2}} + p_{t-2})$ in view of the fact that $y^{d_{t-1}} + p_{t-1}$ and μ_{t-1} are coprime. Repeating this argument, we find that the gcd of $f_0(0, y), \dots, f_t(0, y)$ is 1, which implies that $p_x(I_\lambda) = k[y]$.

(b) Conversely, suppose that $y^{d_j} + p_j$ and μ_j are not coprime for some j , that is, $(\mu_j, y^{d_j} + p_j) \neq k[y]$. By (5.3) and Lemma 5.4, $f_0(0, y), \dots, f_{j-1}(0, y)$ belong to the ideal $(\mu_j, y^{d_j} + p_j)$. On the other hand, again by (5.3), the remaining polynomials $f_j(0, y), \dots, f_t(0, y)$ are divisible by $y^{d_j} + p_j$ and hence belong to $(\mu_j, y^{d_j} + p_j)$. Therefore, $p_x(I_\lambda) \subseteq (\mu_j, y^{d_j} + p_j) \neq k[y]$. \square

For the proof of Theorem 5.1, we also need the following result.

LEMMA 5.6. *If $y^{d_j} + p_j$ and μ_j are coprime for all $j > i$, then the polynomials $q_{i+1,i}, \dots, q_{t+1,i}$ of (5.2) are coprime.*

Proof. Note that the polynomials $q_{r,s}$ are defined for $r > s$ in (5.2). Proceeding as in Part (a) of the proof of Proposition 5.5 and using (5.2), we show by descending induction on j that the gcd of $q_{j+1,i}, \dots, q_{t+1,i}$ is

$$(y^{d_{i+1}} + p_{i+1}) \cdots (y^{d_j} + p_j).$$

In particular, for $j = i + 1$, the gcd of $q_{i+2,i}, \dots, q_{t+1,i}$ is $(y^{d_{i+1}} + p_{i+1})$. The conclusion follows from this fact together with the coprimality of $(y^{d_{i+1}} + p_{i+1})$ and of $q_{i+1,i} = \mu_{i+1}$. \square

5.3. Proof of Theorem 5.1

We have to count the possible matrices M_λ such that M_λ^x and M_λ^y are invertible; equivalently, to count the matrices M_λ^x since x in M_λ appears with constant coefficients. By Propositions 4.1 and 5.5, it suffices to count the matrices M_λ^x over $\mathbb{F}_q[y]$ such that $p_i(0) \neq 0$ and $y^{d_i} + p_i$ and μ_i are coprime for all $i = 1, \dots, t$. We consider these conditions successively for $i = t, t - 1, \dots, 1$.

Assume first that all integers d_1, \dots, d_t are nonzero. For $i = t$, $y^{d_t} + p_t$ is a monic polynomial of degree d_t with nonzero constant term, $\mu_t = p_{t+1,t}$ is of degree $< d_t$, and both polynomials are coprime. It follows from Lemma 2.5 (or from

Proposition 2.3 with $d = d_t$ and $h = 1$) that we have $(q - 1)^2(q^{2d_t} - 1)/(q^2 - 1)$ possible choices for the last column of M_λ .

For $i = t - 1$, it follows from (5.1) that $\mu_{t-1} = P_1 Q_1 + P_2 Q_2$, where $Q_1 = q_{t,t-1}$ and $Q_2 = -q_{t+1,t-1}$, which are coprime by Lemma 5.6, $P_1 = p_{t,t-1}$ and $P_2 = p_{t+1,t-1}$, which are both polynomials of degree $< d_{t-1}$. The polynomial $P = y^{d_{t-1}} + p_{t-1}$ is monic of degree d_{t-1} with nonzero constant term, and $Q = \mu_{t-1} = P_1 Q_1 + P_2 Q_2$ is coprime to P by the coprimality condition. It then follows from Proposition 2.3 applied to the case $d = d_{t-1}$ and $h = 2$ that there are

$$(q - 1)^2 q^{d_{t-1}} \frac{q^{2d_{t-1}} - 1}{q^2 - 1}$$

possible choices for the $(t - 1)$ th column of M_λ .

In general, the polynomial $P = y^{d_i} + p_i$ is monic of degree d_{i-1} with nonzero constant term and is assumed to be coprime to $Q = \mu_i = \sum_{j=1}^{t-i+1} p_{i+j,i} q_{i+j,i}$. By Lemma 5.6 the polynomials $q_{i+1,i}, \dots, q_{t+1,i}$ are coprime. Applying Proposition 2.3 to the case $d = d_i$ and $h = t + 1 - i$, we see that there are

$$(q - 1)^2 q^{(t-i)d_i} \frac{q^{2d_i} - 1}{q^2 - 1}$$

possible choices for the i th column of M_λ^x .

In the end we obtain a number of possible entries for M_λ^x equal to

$$\prod_{i=1}^t (q - 1)^2 q^{(t-i)d_i} \frac{q^{2d_i} - 1}{q^2 - 1} = q^{n - \ell(\lambda)} \prod_{i=1}^t (q - 1)^2 \frac{q^{2d_i} - 1}{q^2 - 1}$$

since $\ell(\lambda) = \sum_{i=1}^t d_i$ and $n = |\lambda| = \sum_{i=1}^t (t - i + 1)d_i$. We have thus proved the theorem when d_1, \dots, d_t are all nonzero.

Let E be the subset of $\{1, \dots, t\}$ consisting of those subscripts i for which $d_i = 0$. (Note that 1 does not belong to E since $d_1 > 0$.) Assume now that E is nonempty and set $t' = t - \text{card } E$. By assumption $t' < t$. For any positive integer $i \leq t'$, let d'_i be equal to the i th nonzero d_i . The integers $d'_1 = d_1, d'_2, \dots, d'_{t'}$ are positive.

Recall that if $i \in E$, then the i th column of the matrix M_λ^x is zero except for the (i, i) -entry, which is 1. Permuting rows and columns, we may rearrange M_λ^x into a matrix M'_λ of the form

$$M'_\lambda = \begin{pmatrix} M'_v & 0 \\ N & I_{t-t'} \end{pmatrix},$$

where $I_{t-t'}$ is the identity matrix of size $(t - t')$. The $(t' + 1) \times t'$ -matrix M'_v has the same form as M_λ^x with t replaced by t' , the sequence d_1, \dots, d_t by the shorter sequence $d'_1, \dots, d'_{t'}$, and the partition λ by the partition ν associated with the sequence $d'_1, \dots, d'_{t'}$.

Let f'_i be the determinant of the square matrix obtained from M'_λ by deleting its $(i + 1)$ th row. It is clear that up to sign and to reordering the maximal minors

$f'_0, \dots, f'_{t'}$ of M'_λ are the same as those of M_λ . In view of the special form of M'_λ , observe that

$$f'_i = \begin{cases} f_i^{(v)} & \text{if } 0 \leq i \leq t', \\ 0 & \text{if } t' < i \leq t, \end{cases}$$

where $f_i^{(v)}$ is the determinant of the $t' \times t'$ -matrix obtained from M_v by deleting its $(i + 1)$ th row.

The number of possible entries of M_λ , which is the same as the number of possible entries of M'_λ , is then the product of the number of possible entries of N , which is a power of q , and of the number of possible entries of M_v . Since $d'_1, \dots, d'_{t'}$ are positive, by the first part of the proof, we know that the number of possible entries of M_v is the product of a power of q by

$$\prod_{i=1}^{t'} (q - 1)^2 \frac{q^{2d'_i} - 1}{q^2 - 1}.$$

In other words, the number of possible entries of M_λ is

$$q^c \prod_{\substack{i=1, \dots, t \\ d_i \geq 1}} (q - 1)^2 \frac{q^{2d_i} - 1}{q^2 - 1}$$

for some nonnegative integer c . Now since the invertible Gröbner cell $C_\lambda^{x,y}$ is a Zarisky open subset of the affine Gröbner cell C_λ , the degree of the previous polynomial in q must be the same as the degree of the cardinal of C_λ , which is $q^{n+\ell(\lambda)}$ by Section 3.1. This suffices to establish that $c = n - \ell(\lambda)$ and to complete the proof of the theorem.

5.4. Proof of Theorem 1.1

By our remark at the beginning of Section 5, the number $C_n(q)$ of ideals of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$ of codimension n is given by

$$C_n(q) = \sum_{\lambda \vdash n} \text{card } C_\lambda^{x,y}, \tag{5.4}$$

where $C_\lambda^{x,y}$ is the *invertible Gröbner cell* associated with the partition λ . The equality in Theorem 1.1 follows then from the formula for $\text{card } C_\lambda^{x,y}$ given in Theorem 5.1.

By Corollary 5.2 (a) $\text{card } C_\lambda^{x,y}$ is a monic polynomial of degree $n + \ell(\lambda)$ with integer coefficients. Therefore, $C_n(q)$ has integer coefficients, and its degree is $\max\{n + \ell(\lambda) \mid \lambda \vdash n\}$. Now $\ell(\lambda)$ is maximal if and only if $\lambda = 1^n$, in which case $\ell(\lambda) = n$. Therefore $C_n(q)$ is monic, and its degree is $2n$.

Since $v(\lambda) \geq 1$, it follows from the formula in Theorem 5.1 that $\text{card } C_\lambda^{x,y}$ is divisible by $(q - 1)^2$ for each invertible Gröbner cell. Therefore, the polynomial $C_n(q)$ is divisible by $(q - 1)^2$.

6. Proofs of the Corollaries

We now start the proofs of Corollaries 1.2 and 1.4.

6.1. Proof of Corollary 1.2

Since $C_n(q)$ and $(q - 1)^2$ are both monic with integer coefficients, so is $P_n(q)$. The latter is the sum over all partitions of n of the polynomials $P_\lambda(q)$ (introduced in Corollary 5.2(b)). By Corollary 5.2(c)–(d), we have $P_\lambda(1) = 0$ if $v(\lambda) \geq 2$ and if $v(\lambda) = 1$, then λ is of the form t^d , where $dt = n$, in which case $P_\lambda(1) = d$. The desired formula for $P_n(1)$ follows.

6.2. Proof of Corollary 1.4

As in the proof of Theorem 4.6, we consider each partition λ as a union of rectangular partitions i^{e_i} , with e_i parts of length i , for $e_i \geq 1$ and distinct $i \geq 1$. Recall that $|\lambda| = \sum_i i e_i$, $\ell(\lambda) = \sum_i e_i$, and $v(\lambda) = \sum_i 1$. To indicate the dependance of e_i on λ , we write $e_i = e_i(\lambda)$. We then obtain the following statement.

PROPOSITION 6.1. *Let s_1, s_2, \dots be new variables. We have the infinite product expansion*

$$1 + \sum_{\lambda} \text{card } C_{\lambda}^{x,y} s_1^{e_1(\lambda)} s_2^{e_2(\lambda)} \dots = \prod_{i \geq 1} \frac{(1 - q^i s_i)^2}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}.$$

Proof. Proceeding as in the proof of Theorem 4.6 and using Theorem 5.1, we deduce that the left-hand side is equal to

$$1 + \sum_{\lambda} \prod_{i \geq 1} (q - 1)^2 \frac{q^{2e_i(\lambda)} - 1}{q^2 - 1} q^{ie_i(\lambda) - e_i(\lambda)} s_i^{e_i(\lambda)},$$

which in turn is equal to

$$\begin{aligned} & \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \sum_{e_i \geq 1} ((q^{i+1} s_i)^{e_i} - (q^{i-1} s_i)^{e_i}) \right) \\ &= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \left(\frac{q^{i+1} s_i}{1 - q^{i+1} s_i} - \frac{q^{i-1} s_i}{1 - q^{i-1} s_i} \right) \right) \\ &= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2}{q^2 - 1} \frac{(q^2 - 1) q^{i-1} s_i}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)} \right) \\ &= \prod_{i \geq 1} \left(1 + \frac{(q - 1)^2 q^{i-1} s_i}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)} \right) \\ &= \prod_{i \geq 1} \frac{(1 - q^i s_i)^2}{(1 - q^{i+1} s_i)(1 - q^{i-1} s_i)}. \end{aligned}$$

□

Proof of Corollary 1.4. (a) Replace s_i by $(t/q)^i$ in Proposition 6.1, use (5.4) and Theorem 1.1, and observe that $(1 - qt^i)(1 - q^{-1}t^i) = 1 - (q + q^{-1})t^i + t^{2i}$.

(b) The infinite product is clearly invariant under the transformation $q \leftrightarrow q^{-1}$; thus, $C_n(q^{-1}) = q^{-2n}C_n(q)$. Together with $\deg C_n(q) = 2n$, this implies that $C_n(q)$ is palindromic. The polynomial $P_n(q)$ is palindromic as a quotient of two palindromic polynomials.

6.3. An Alternative Proof of Corollary 1.4(a)

After we made public a first version of this article, we learnt of an alternative geometric approach to the polynomials $C_n(q)$. Indeed, Göttsche and Soergel determined the mixed Hodge structure of the punctual Hilbert schemes of any smooth complex algebraic surface (see [15, Thm. 2]). Applying their result to the Hilbert scheme $H_{\mathbb{C}}^n = \text{Hilb}^n(\mathbb{C}^{\times} \times \mathbb{C}^{\times})$ of n points of the complex two-dimensional torus, Hausel, Letellier, and Rodriguez-Villegas observed in [20, Thm. 4.1.3] that the compactly supported mixed Hodge polynomial $H_c(H_{\mathbb{C}}^n; q, u)$ of $H_{\mathbb{C}}^n$ fits into the equality of formal power series

$$1 + \sum_{n \geq 1} H_c(H_{\mathbb{C}}^n; q, u) \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 + u^{2i+1}t^i)^2}{(1 - u^{2i+2}qt^i)(1 - u^{2i}q^{-1}t^i)}. \tag{6.1}$$

Setting $u = -1$ in (6.1), we obtain an infinite product expansion for the generating function of the E -polynomial $E(H_{\mathbb{C}}^n; q) = H_c(H_{\mathbb{C}}^n; q, -1)$ of $H_{\mathbb{C}}^n$, namely

$$1 + \sum_{n \geq 1} E(H_{\mathbb{C}}^n; q) \frac{t^n}{q^n} = \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}}. \tag{6.2}$$

Now, $H_{\mathbb{C}}^n$ is strongly polynomial-count in the sense of Nick Katz (see [21, Appendix]), probably a well-known fact (which also follows from the computations in the present paper). Therefore, by [21, Thm. 6.1.2] the E -polynomial counts the number of elements of H^n over the finite field \mathbb{F}_q , which is also the number $C_n(q)$ of ideals of codimension n of $\mathbb{F}_q[x, y, x^{-1}, y^{-1}]$. Thus (6.2) implies the equality of Corollary 1.4(a).

REMARK 6.2. In the same vein as before, there is a geometric explanation of the palindromicity of the polynomials $C_n(q)$. De Cataldo, Hausel, and Migliorini [9] observed that any diffeomorphism between $\mathbb{C}^{\times} \times \mathbb{C}^{\times}$ and the cotangent bundle $E \times \mathbb{C}$ of the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$ induces a linear isomorphism of graded vector spaces between the cohomology groups of the corresponding Hilbert schemes: $H^*(H_{\mathbb{C}}^n, \mathbb{Q}) \cong H^*(\text{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$. This isomorphism does not preserve the mixed Hodge structures, as that on the right-hand side is pure, whereas that on the left-hand side is not. Nevertheless, such an isomorphism identifies the weight filtration on $H^*(H_{\mathbb{C}}^n, \mathbb{Q})$ with the perverse Leray filtration on $H^*(\text{Hilb}^n(E \times \mathbb{C}), \mathbb{Q})$ associated with the natural projective map from $\text{Hilb}^n(E \times \mathbb{C})$ to the n th symmetric product of \mathbb{C} induced by the projection of $E \times \mathbb{C}$ on the second factor. The perverse Leray filtration is “palindromic” as

a consequence of the relative hard Lefschetz theorem for that map (see [9, Thms. 4.1.1 and 4.3.2]).

Note that Hausel, Letellier, and Rodriguez-Villegas observed a similar palindromicity for the E -polynomial of certain character varieties and termed it “curious Poincaré duality” in [19, Cor. 5.2.4] (see also [21, Cor. 3.5.3] and [18, Cor. 1.4]).

REMARK 6.3. The natural action of the group $\mathbb{C}^\times \times \mathbb{C}^\times$ on itself induces an action on the Hilbert scheme $H_{\mathbb{C}}^n$. Consider the GIT quotient $\tilde{H}_{\mathbb{C}}^n = H_{\mathbb{C}}^n / (\mathbb{C}^\times \times \mathbb{C}^\times)$. Using [21, Thm. 2.2.12] and [19, Section 5.3], we see that the E -polynomial of $\tilde{H}_{\mathbb{C}}^n$ is given by

$$E(\tilde{H}_{\mathbb{C}}^n; q) = E(H_{\mathbb{C}}^n; q) / (q - 1)^2 = C_n(q) / (q - 1)^2 = P_n(q).$$

Recall from Introduction (see also Appendix) that the coefficients of $P_n(q)$ are all nonnegative. Therefore, $\tilde{H}_{\mathbb{C}}^n$ provides an example of a polynomial-count variety with odd cohomology and a counting polynomial with nonnegative coefficients. This implies nontrivial cancellation for the mixed Hodge numbers of $\tilde{H}_{\mathbb{C}}^n$. No similar positivity phenomenon was observed for the character varieties investigated by Hausel, Letellier, and Rodriguez-Villegas.

6.4. Proof of Corollary 1.5

It is well known that the generating function of the partition function $p(n)$ is

$$\sum_{n \geq 0} p(n)t^n = \prod_{i \geq 1} \frac{1}{1 - t^i}.$$

By [1, Eq. (1.11) and Thm. 1] the generating function for the integers $N_V(m, n)$ is

$$\sum_{m \in \mathbb{Z}} \sum_{n \geq 0} N_V(m, n)q^m t^n = \prod_{i \geq 1} \frac{1 - t^i}{1 - (q + q^{-1})t^i + t^{2i}}.$$

Thus by Corollary 1.4 we have

$$\begin{aligned} \left(\sum_{n \geq 0} p(n)t^n \right) \left(1 + \sum_{n \geq 1} \frac{C_n(q)}{q^n} t^n \right) &= \prod_{i \geq 1} \frac{1}{1 - t^i} \prod_{i \geq 1} \frac{(1 - t^i)^2}{1 - (q + q^{-1})t^i + t^{2i}} \\ &= \prod_{i \geq 1} \frac{1 - t^i}{1 - (q + q^{-1})t^i + t^{2i}} \\ &= \sum_{m \in \mathbb{Z}} \sum_{n \geq 0} N_V(m, n)q^m t^n. \end{aligned}$$

The desired formula follows immediately.

Appendix. The Coefficients of the Polynomials $C_n(q)$ and $P_n(q)$

We now state the results of the companion paper [22] on the coefficients of the polynomials $C_n(q)$ and $P_n(q)$.

Since $C_n(q)$ and $P_n(q)$ are palindromic of respective degrees $2n$ and $2n - 2$, we may expand $C_n(q)$ and $P_n(q)$ as follows:

$$C_n(q) = c_{n,0}q^n + \sum_{i=1}^n c_{n,i}(q^{n+i} + q^{n-i}),$$

where $c_{n,0}, c_{n,1}, c_{n,2} \dots$ are integers, and

$$P_n(q) = a_{n,0}q^{n-1} + \sum_{i=1}^{n-1} a_{n,i}(q^{n+i-1} + q^{n-i+1}),$$

where $a_{n,0}, a_{n,1}, a_{n,2} \dots$ are integers.

By Theorem 1.1 of [22] the coefficients $c_{n,i}$ of $C_n(q)$ are given by the following formulas: (a) For the central coefficients $c_{n,0}$, we have

$$c_{n,0} = \begin{cases} 2(-1)^k & \text{if } n = k(k+1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) For the noncentral coefficients ($i \geq 1$), we have

$$c_{n,i} = \begin{cases} (-1)^k & \text{if } n = k(k+2i+1)/2 \text{ for some integer } k \geq 1, \\ (-1)^{k-1} & \text{if } n = k(k+2i-1)/2 \text{ for some integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that in Item (b) the first two conditions are mutually exclusive.

As for the coefficients of $P_n(q)$, the coefficient $a_{n,i}$ is by [22, Thm. 1.2] equal to the number of divisors d of n such that

$$\frac{i + \sqrt{2n + i^2}}{2} < d \leq i + \sqrt{2n + i^2}.$$

It follows that all coefficients $a_{n,i}$ of $P_n(q)$ are nonnegative integers.

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