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A GENERALIZATION OF SOLOMON'S ALGEBRA FOR HYPEROCTAHEDRAL GROUPS AND OTHER WREATH PRODUCTS

R. Mantaci[*]

Università di Roma 'La Sapienza' and L.I.R. Université Rouen

and C. Reutenauer[†]

Université du Québec à Montréal.

Abstract

In this paper we give a combinatorial rule to compute the composition of two convolution products of endomorphisms of a free associative algebra and deduce the construction of a subalgebra of $\mathbb{Q}\mathcal{B}_n$ (the group algebra of Hyperoctahedral group) which contains the descent algebra $\Sigma\mathcal{B}_n$. We also deduce a proof of the multiplication rule in the algebra $\Sigma\mathcal{B}_n$. Finally, we generalize this construction to other wreath products of symmetric groups by abelian groups.

INTRODUCTION

Since the appearance of the fundamental paper of Solomon [So], it has been a well-known fact that the group algebra of any Coxeter group G contains a remarkable subalgebra called the *descent Algebra*. The reader who wants to know more about Coxeter groups will be able to find a completely detailed basic presentation of their general theory in [Bo].

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The result of Solomon has been put in some simpler forms by Moszkowski [Mo84], who also extended it to submodules of the group algebra [Mo89]. Another proof of Solomon's result has been given in the case of $\mathcal{S}_n$ by Garsia and Remmel [G-R] and Gessel [Ge].

Other authors took an interest in decomposing the multiplicative structure of descent algebras. For instance, Garsia and the second author gave a decomposition in the case in which G is the symmetric group $\mathcal{S}_n$ (Coxeter group of type $\mathcal{A}_{n-1}$), exploiting the action of the symmetric group on the free Lie algebra ([G-Re]), whereas F. and N. Bergeron in [B-B] showed that a similar decomposition is valid also for the Hyperoctahedral group $\mathcal{B}_n$.

The group $\mathcal{B}_n$ being the wreath product of $\mathcal{S}_n$ by $\mathcal{C}_2$, the cyclic group of order 2, we have looked for a possible generalization of these results to the wreath products $\mathcal{S}_n[\mathcal{C}_p]$, and we constructed an algebra which is more general than the Solomon's algebra also if $p = 2$. Indeed, refining the definition of descent classes, we constructed a subalgebra of $\mathbb{Q}\mathcal{B}_n$ having dimension $2 \cdot 3^{n-1}$ which contains $\Sigma\mathcal{B}_n$, the descent algebra of $\mathcal{B}_n$. WE remark that an analoguous construction can be done for groups $\mathcal{S}_n[G]$ where G is any abelian group.

This definition presents at least three advantages compared to the one of the algebra $\Sigma\mathcal{B}_n$: first, this algebra has an interpretation in terms of the action of the Hyperoctahedral group on a suitable free associative algebra, (what led us to generalize the combinatorial rule to compute the composition of two convolution products of endomorphisms of a free associative algebra), secondly it lends itself to a generalization to the other wreath products and finally the multiplication rule in this algebra is much easier than in $\Sigma\mathcal{B}_n$.

We organized this work in seven sections.

Section 1 contains the necessary preliminaries and notations.

In section 2 we establish the result from which all the others are deduced, we show indeed that the combinatorial rule to decompose products of the type :

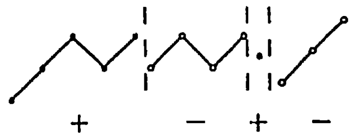
$$(q_{h_1} * q_{h_2} * \cdots * q_{h_k}) \circ (q_{k_1} * q_{k_2} * \cdots * q_{k_r})$$

(where q_i are the projections of a free associative algebra $\mathbb{Q}\langle A\rangle$ over $\mathbb{Q}\langle A\rangle_i$, its homogeneous component of degree i), applies also to the case in which, instead of the projections q_i one takes the restrictions of arbitrary degree-preserving algebra endomophisms of $\mathbb{Q}\langle A\rangle$ to its subspaces $\mathbb{Q}\langle A\rangle_i$.

As first application, we show in Section 3 that the subspace $\Omega\mathcal{B}_n$ of $\mathbb{Q}\mathcal{B}_n$ spanned by the elements that we call *shape classes*, (refinements of descent classes) is a subalgebra of $\mathbb{Q}\mathcal{B}_n$ having dimension $2*3^{n-1}$ and containing $\Sigma\mathcal{B}_n$.

Shape classes are the sum of all permutations having the same sign pattern and, inside each segment of integers having the same sign, the same up-down pattern. Shape classes can be indexed by signed composition of the integer n, that is, sequences of integers $c_1 c_2 \ldots c_k$ such that $\sum_{j=1}^{k} |c_j| = n$. For instance, the class $S_{=C}$ corresponding to the signed composition

$C = 3\ 2\ \bar{2}\ \bar{2}\ 1\ \bar{3}$ contains all the permutations σ of $\mathcal{B}_{13}$ such that only $\sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{11}, \sigma_{12}, \sigma_{13}$ are negatives (barred), and such that in the segments of elements of the same sign the up-down patterns of the absolute value of the integers are :



In section 3 we also show that the multiplication rule in $\Omega\mathcal{B}_n$ can be easily given in a combinatorial way : if one takes as generators the elements S_C, sum of all the $S_{=H}$ such that the signed permutation H is less fine than the signed permutation C, then the product of two classes S_C and S_D is the sum of those classes $S_{C(M)}$, whose composition $C(M)$ is obtained from some suitable matrices that we call *compatible* with the compositions C and D.

In section 4 we give a further generalization of the main result of Section 2. We show indeed that one can suppose that the endomorphisms involved in the convolution products are possibly composed with the application $\tilde{}$, which acts on the words by reversing the order of their letters. A combinatorial rule is given also in this case.

Using the latest generalization, we deduce in section 5 the multiplication rule in the algebra $\Sigma\mathcal{B}_n$ (this proof had just been sketched in [B-B]), we give this way an explicit relation between the two algebras $\Omega\mathcal{B}_n$ and $\Sigma\mathcal{B}_n$.

In Section 6, we generalize the construction of the algebra $\Omega\mathcal{B}_n$ to all the wreath products $\mathcal{S}_n[\mathcal{C}_p]$ whose elements are represented as multi-signed permutations ; this way we obtain a subalgebra of $\mathbb{Q}(\mathcal{S}_n[\mathcal{C}_p])$ having dimension $p\cdot(p+1)^{n-1}$. Shape classes, indexed by *multi-signed compositions*, are multiplied always with the same rule (the signs of the integers in the matrices are obtained by summing modulo p the corresponding signs of the integers in the compositions).

1. PRELIMINARIES

Let A be an alphabet, and let $\mathbb{Q}\langle A\rangle$ be the free associative algebra over A, the algebra of all linear combinations of words in A. The algebra $\mathbb{Q}\langle A\rangle$ is a graded algebra, this means that, if $\mathbb{Q}\langle A\rangle_n$ denotes the space of all homogeneous polynomial of degree n, then we have :

$$\mathbb{Q}\langle A\rangle = \bigoplus_{n\geq 0} \mathbb{Q}\langle A\rangle_n .$$

In $End_{\mathbb{Q}}\mathbb{Q}\langle A\rangle$, the algebra of linear endomorphisms of $\mathbb{Q}\langle A\rangle$, is defined an associative product called *convolution*. To define it, we need to introduce the following homomorphism of

$\mathbb{Q}$-algebras. Let $\delta: \mathbb{Q}\langle A\rangle \rightarrow \mathbb{Q}\langle A\rangle \otimes_{\mathbb{Q}} \mathbb{Q}\langle A\rangle$, such that. $\delta(a) = a \otimes 1 + 1 \otimes a$, for any letter a. We also need to define a linear mapping $conc : \mathbb{Q}\langle A\rangle \otimes \mathbb{Q}\langle A\rangle \rightarrow \mathbb{Q}\langle A\rangle$ by : $conc(P\otimes Q) = PQ$. Finally, the convolution product, denoted by $*$, is defined by $f*g = conc \circ (f \otimes g) \circ \delta$.

Define more generally an homomorphism of algebras δ_p: $\mathbb{Q}\langle A\rangle \rightarrow \mathbb{Q}\langle A\rangle^{\otimes p}$ by :

$$\delta_p(a) = a \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes a \otimes \ldots \otimes 1 + \ldots + 1 \otimes 1 \otimes \ldots \otimes a \quad \text{for any letter } a.$$

(in this sum each tensor product has exactly p factors)

Indeed, δ_p extends to a unique algebra homomorphism $\mathbb{Q}\langle A\rangle \rightarrow \mathbb{Q}\langle A\rangle^{\otimes p}$, this extension is homogeneous and degree-preserving, note that $\delta = \delta_2$. The following result shows that δ_p has another equivalent definition, in terms of the shuffle product, which is denoted by ⧢ .
We remember that the shuffle product of two words u and v of A^* is the associative operation whose result is the polynomial :

$$u ⧢ v = \sum h$$

with $h = u_1v_1u_2v_2\ldots u_nv_n$, $n \geq 0$, u_i, $v_i \in A^*$, $u = u_1u_2\ldots u_n$ and $v = v_1v_2\ldots v_n$.

PROPOSITION 1.1. *For each word w, one has :*

$$\delta_p(w) = \sum_{u_1 \ldots u_p \in A^*} (u_1 ⧢ u_2 ⧢ \ldots ⧢ u_p, w)\, u_1 \otimes \ldots \otimes u_p \qquad (1.1)$$

where(P, w) represents the coefficient of the monomial w in the polynomial P.

Then, for any endomorphisms $f_1, \ldots, f_p$, their convolution is :

$$f_1 * \ldots * f_p = conc_p \circ (f_1 \otimes \ldots \otimes f_p) \circ \delta_p \,. \qquad (1.2)$$

where $conc_p\,(u_1 \otimes \ldots \otimes u_p) = u_1 \ldots u_p$.

2. THE MAIN RESULT

Let α be an endomorphism of the algebra $\mathbb{Q}\langle A\rangle$ which is degree preserving, in other terms, α transforms letters of the alphabet A into linear combinations of letters.

We denote by $\alpha|_n$ the linear map which coincides with α over the subspace $\mathbb{Q}\langle A\rangle_n$ and which is equal to zero over the spaces $\mathbb{Q}\langle A\rangle_m$ when $m \neq n$. In other terms, $\alpha|_n$ is the linear map which coincides with $q_n \circ \alpha = \alpha \circ q_n$ where q_n is the projection over the subspace $\mathbb{Q}\langle A\rangle_n$.

DEFINITIONS 2.1. If n is a positive integer we call *pseudo-composition* of n a sequence H of nonnegative integers $h_1 h_2 \ldots h_k$ such that $\sum_{j=1}^{k} h_j = n$. The *weight* of the pseudo composition is $|H| = n$, its *length* is $\ell(H) = k$. The integers $h_1, h_2, \ldots, h_k$ are called *parts* of H. A *composition* is a pseudo-composition in which $h_j > 0$ for all j. To each pseudo-composition H is naturally associated the composition $C(H)$ obtained by removing the 0's in H.

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be degree-preserving algebra endomorphisms of $\mathbb{Q}<A>$.

NOTATION. If $H = h_1 h_2 \ldots h_k$ is a pseudo-composition of the integer n, we denote by $(\alpha_1,\alpha_2, \ldots,\alpha_k)_H$ the convolution product :

$$(\alpha_1|_{h_1} * \alpha_2|_{h_2} * \ldots * \alpha_k|_{h_k}),$$

and by $(\alpha_1,\alpha_2, \ldots,\alpha_k)_{\otimes H}$ the tensor product :

$$(\alpha_1|_{h_1} \otimes \alpha_2|_{h_2} \otimes \ldots \otimes \alpha_k|_{h_k}).$$

Observe that for all endomorphisms α, the map $\alpha|_0$ is the endomorphism which sends each polynomial onto its constant term. Hence $\alpha|_0$ is the neutral element for the convolution.

If $L = l_1 l_2 \ldots l_p$ is another composition, and $\beta_1, \beta_2, \ldots, \beta_p$ other degree-preserving endomorphisms, we want to give a formula to compute the composition :

$$(\alpha_1,\alpha_2, \ldots,\alpha_k)_H \circ (\beta_1, \beta_2, \ldots, \beta_p)_L . \qquad (2.1)$$

In particular, when we take $\alpha_i = \beta_j = identity$ for all i and j, we have $\alpha_i|_n = \beta_j|_n = q_n$ where q_n is the map equal to the identity over $\mathbb{Q}<A>_n$ and sending $\mathbb{Q}<A>_m$ onto 0 if $m \neq n$.

The linear span of the elements $q_H = q_{h_1} * q_{h_2} * \ldots * q_{h_k}$ is the Convolution Algebra Γ studied in [Re]. The product :

$$(q_{h_1} * q_{h_2} * \ldots * q_{h_k}) \circ (q_{l_1} * q_{l_2} * \ldots * q_{l_p})$$

is a linear combination of elements of the same type with integer coefficients, and these coefficients have a combinatorial interpretation (this is essentially equivalent to the multiplication rule in [G-R]). We will show that similar arguments hold for the more general composition (2.1).

Let us introduce some other notations and terminology.

Given a matrix $M = (m_{ij})$ $1 \le i \le k$, $1 \le j \le p$, of nonnegative integers, its *row sum* is the pseudo- composition :

$$(m_{11} + m_{12} + \ldots + m_{1p}, m_{21} + m_{22} + \ldots + m_{2p}, \ldots , m_{k1} + m_{k2} + \ldots + m_{kp}) .$$

Similarly, its *column sum* is the pseudo composition :

$$(m_{11}+m_{21}+\ldots+m_{k1}, m_{12}+m_{22}+\ldots+m_{k2}, \ldots, m_{1p}+m_{2p}+\ldots+m_{kp}) .$$

The *pseudo-composition associated to M* is :

$$C(M) = (m_{11}, m_{12}, \ldots, m_{1p}, m_{21}, m_{22}, \ldots, m_{2p}, \ldots, m_{k1}, m_{k2}, \ldots, m_{kp}) .$$

In other words, $C(M)$ is the pseudo-composition obtained by reading the integers of M, row by row from left to right.

EXAMPLE. $M = \begin{vmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix}$.

The row sum of M is 4 3, its column sum is 2 2 3, and its associated pseudo-composition is 1 0 3 1 2 0.

NOTE. If M is a matrix, H its row sum and L its column sum, then we will say that M is *compatible* with H and L (in *this* order).

We can now state the main theorem of this section :

THEOREM 2.2. *Let $\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_p$ be degree-preserving endomorphisms of the free associative algebra $\mathbb{Q}\langle A\rangle$, $H=h_1\, h_2 \ldots h_k$ and $L=l_1\, l_2 \ldots l_p$ be two composition of the integer n, then the composition product :*

$$(\alpha_1,\alpha_2, \ldots,\alpha_k)_H \circ (\beta_1, \beta_2, \ldots, \beta_p)_L$$

is a sum of all the elements :

$$\begin{pmatrix} (\alpha_1\circ\beta_1), (\alpha_1\circ\beta_2), \ldots, (\alpha_1\circ\beta_p), \\ (\alpha_2\circ\beta_1), (\alpha_2\circ\beta_2), \ldots, (\alpha_2\circ\beta_p), \\ \ldots \\ (\alpha_k\circ\beta_1), (\alpha_k\circ\beta_2), \ldots, (\alpha_k\circ\beta_p) \end{pmatrix}_{C(M)}$$

such that the matrix $M = (m_{ij})$ has row sum equal to H and column sum equal to L (or, it is compatible with H and L).

EXAMPLE. Let H = 2 2, L = 3 1. Then the compatible matrices are :

$$\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \text{ and } \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}.$$

Thus, $(\alpha_1,\alpha_2)_{2\,2} \circ (\beta_1, \beta_2)_{3\,1} =$

$=\Big((\alpha_1\circ\beta_1),(\alpha_1\circ\beta_2),(\alpha_2\circ\beta_1),(\alpha_2\circ\beta_2)\Big)_{1\,1\,2\,0}+\Big((\alpha_1\circ\beta_1),(\alpha_1\circ\beta_2),(\alpha_2\circ\beta_1),(\alpha_2\circ\beta_2)\Big)_{2\,0\,1\,1}$

$=(\alpha_1\circ\beta_1)|_1*(\alpha_1\circ\beta_2)|_1*(\alpha_2\circ\beta_1)|_2 + (\alpha_1\circ\beta_1)|_2*(\alpha_2\circ\beta_1)|_1*(\alpha_2\circ\beta_2)|_1.$

We call a subalgebra of $End_{\mathbb{Q}}\mathbb{Q}\langle A\rangle$ relatively to * *a convolution subalgebra.*

COROLLARY 2.3. *Let M be a subalgebra of the algebra of degree preserving endomorphisms of* $\mathbb{Q}\langle A\rangle$. *Then the convolution subalgebra generated by the* $\alpha|_n$, $\alpha\in M$, $n\geq 0$ *is closed under composition and hence is a subalgebra of* $End_{\mathbb{Q}}\mathbb{Q}\langle A\rangle$ *relatively to* + *and* $\circ$.

To prove Theorem 2.2 we will follow the same line of proof as in the case of Convolution Algebra, showing that all results hold true when one makes the computations for any endomorphism which is degree-preserving. Most of the necessary lemmas remain exactly the same so we will state them without proof. The reader will find these proofs in [Re].

LEMMA 2.4. *One has the formula*

$$\delta_k\circ\alpha|_l = \sum_{h_1+\ldots+h_k=l} (\alpha|_{h_1}\otimes\alpha|_{h_2}\otimes\ldots\otimes\alpha|_{h_k})\circ\delta_k.$$

Proof. We first verify that :

$$\delta_k\circ\alpha = \alpha^{\otimes k}\circ\delta_k.$$

On both sides we have algebra homomorphisms, so it suffices to check the action of the two maps over the letters of A. Let a be a letter of A then $\alpha(a)$ is a linear combination of letters, hence,

$\delta_k\circ\alpha(a) =$

$=\alpha(a)\otimes 1\otimes\ldots\otimes 1 + 1\otimes\alpha(a)\otimes\ldots\otimes 1+\ldots+1\otimes 1\otimes\ldots\otimes\alpha(a) =$

$=(\alpha\otimes\alpha\otimes\ldots\otimes\alpha)(a\otimes 1\otimes\ldots\otimes 1+1\otimes a\otimes\ldots\otimes 1+\ldots+1\otimes 1\otimes\ldots\otimes a) =$

$=\alpha^{\otimes k}\circ\delta_k(a)$. (In view of the definition of $\delta_k(a)$).

By homogeneity, both sides vanish when applied to a word of length different from l. So let $w=a_1a_2\ldots a_l$ be a word of length l, we have :

$\delta_k \circ \alpha|_l\,(w) = \delta_k \circ \alpha(w) = \alpha^{\otimes k} \circ \delta_k(w)$ (by the above equality)

$$\Big(\sum_{h\geq 0} \alpha|_h\Big)^{\otimes k} \circ \delta_k(w) = \sum_{h_1+\ldots+h_k} (\alpha|_{h_1} \otimes \alpha|_{h_2} \otimes \ldots \otimes \alpha|_{h_k}) \circ \delta_k(w)$$

which is equal to the right hand side in the claim, by the homogeneity of δ and because w is of length l. □

NOTATION. If $(i_1, \ldots, i_k)$ denotes a sequence of positive integers, all $\leq n$, we denote :

$$conc_{i_1,\ldots,i_k} : \mathbb{Q}\langle A\rangle^{\otimes n} \longrightarrow \mathbb{Q}\langle A\rangle, \quad P_1 \otimes \ldots \otimes P_n \rightarrow P_{i_1} \ldots P_{i_k}.$$

LEMMA 2.5. $\delta_k \circ conc_p = [conc_{1,k+1,\ldots,(p-1)k+1} \otimes \ldots \otimes conc_{k,2k,\ldots,pk}] \circ \delta_k^{\otimes p}$.□

LEMMA 2.6. *For two compositions* $H = h_1, h_2, \ldots, h_p$ *and* $L = l_1, l_2, \ldots, l_p$ *and for algebra endomorphisms* $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_p$, *we have:*

$$(\alpha_1, \alpha_2, \ldots, \alpha_p)_{\otimes H} \circ (\beta_1, \beta_2, \ldots, \beta_p)_{\otimes L} = \begin{cases} (\alpha_1 \circ \beta_1, \alpha_2 \circ \beta_2, \ldots, \alpha_p \circ \beta_p)_{\otimes H} & \text{if } H = L \\ 0 & \text{otherwise} \end{cases}$$

Proof. This is clear. □

LEMMA 2.7. $\delta_k^{\otimes p} \circ \delta_p = \delta_{kp}$.

Proof This follows immediately from Proposition.1.1 and the associativity of the shuffle product. □

For σ in $\mathcal{S}_n$, we denote also by σ the mapping :

$$\mathbb{Q}\langle A\rangle^{\otimes n} \longrightarrow \mathbb{Q}\langle A\rangle^{\otimes n}, P_1 \otimes \ldots \otimes P_n \longrightarrow P_{\sigma(1)} \otimes \ldots \otimes P_{\sigma(n)}.$$

LEMMA 2.8. $conc_{(\sigma 1,\ldots,\sigma n)} = conc_n \circ \sigma$. □

LEMMA 2.9. *For linear functions* $f_1, \ldots, f_n : \mathbb{Q}\langle A\rangle \rightarrow \mathbb{Q}\langle A\rangle$ *and* σ *in* $\mathcal{S}_n$, *one has*

$$\sigma \circ (f_1 \otimes \ldots \otimes f_n) = (f_{\sigma 1} \otimes \ldots \otimes f_{\sigma n}) \circ \sigma.\square$$

LEMMA 2.10. *For σ in $\mathcal{S}_n$, one has :* $\sigma \circ \delta_n = \delta_n$. □

Proof of Theorem 2.2. By definition of convolution product,

$$(\alpha_1,\alpha_2,\ldots,\alpha_k)_H \circ (\beta_1,\beta_2,\ldots,\beta_p)_L$$

$$= conc_k \circ (\alpha_1,\alpha_2,\ldots,\alpha_k)_{\otimes H} \circ \delta_k \circ conc_p \circ (\beta_1,\beta_2,\ldots,\beta_p)_{\otimes L} \circ \delta_p$$

$$= conc_k \circ (\alpha_1,\alpha_2,\ldots,\alpha_k)_{\otimes H} \circ (conc_{1,k+1,\ldots,(p-1)k+1} \otimes \ldots \otimes conc_{k,2k,\ldots,pk}) \circ$$

$$\delta_k^{\otimes p} \circ (\beta_1,\beta_2,\ldots,\beta_p)_{\otimes L} \circ \delta_p \qquad \text{(by Lemma 2.5)}$$

$$= conc_k \circ \Big[\sum_{\substack{n_{11}+\ldots+n_{1p}=h_1 \\ \cdots \\ n_{k1}+\ldots+n_{kp}=h_k}} (conc_{1,k+1,\ldots,(p-1)k+1} \otimes \ldots \otimes conc_{k,2k,\ldots,pk})$$

$$\circ (\alpha_1|_{n_{11}} \otimes \alpha_2|_{n_{21}} \ldots \otimes \alpha_k|_{n_{k1}} \otimes \alpha_1|_{n_{12}} \otimes \ldots \otimes \alpha_k|_{n_{kp}}) \Big]$$

$$\circ \Big[\sum_{\substack{m_{11}+\ldots+m_{k1}=l_1 \\ \cdots \\ m_{1p}+\ldots+m_{kp}=l_p}} \beta_1|_{m_{11}} \otimes \beta_1|_{m_{21}} \ldots \otimes \beta_1|_{m_{k1}} \otimes \beta_2|_{m_{12}} \otimes \ldots \otimes \beta_p|_{m_{kp}} \Big] \circ \delta_k^{\otimes p} \circ \delta_p$$

(because multiplication preserves the degree, and by Lemma 2.4)

$$= conc_k \circ \sum_{\substack{m_{11}+\ldots+m_{1p}=h_1 \\ \cdots \\ m_{k1}+\ldots+m_{kp}=h_k \\ m_{11}+\ldots+m_{k1}=l_1 \\ \cdots \\ m_{1p}+\ldots+m_{kp}=l_p}} (conc_{1,k+1,\ldots,(p-1)k+1} \otimes \ldots \otimes conc_{k,2k,\ldots,pk}) \circ$$

$$\circ ((\alpha_1 \circ \beta_1)|_{m_{11}} \otimes (\alpha_2 \circ \beta_1)|_{m_{21}} \otimes \ldots \otimes (\alpha_k \circ \beta_p)|_{m_{kp}}) \circ \delta_{kp}$$

(by Lemma 2.6 and 2.7)

$$= conc_{1,k+1,\ldots,(p-1)k+1,\ldots,k,2k,\ldots,pk} \circ$$

$$\Big(\sum_{m_{ij}} (\alpha_1 \circ \beta_1)|_{m_{11}} \otimes (\alpha_2 \circ \beta_1)|_{m_{21}} \otimes \ldots \otimes (\alpha_k \circ \beta_p)|_{m_{kp}} \Big) \circ \delta_{kp}$$

(by associativity of concatenation product)

$$= conc_{pk} \circ \sigma \circ \Big(\sum_{m_{ij}} (\alpha_1 \circ \beta_1)|_{m_{11}} \otimes (\alpha_2 \circ \beta_1)|_{m_{21}} \otimes \ldots \otimes (\alpha_k \circ \beta_p)|_{m_{kp}} \Big) \circ \delta_{kp}$$

(where σ is the permutation $\begin{pmatrix} 1 & 2 & \dots & p & & pk-p+1 & pk-p+2 & \dots & pk \\ 1 & k+1 & \dots & (p-1)k+1 & & k & 2k & \dots & pk \end{pmatrix}$)

$$= conc_{pk} \circ \left(\sum_{m_{ij}} (\alpha_1 \circ \beta_1)|_{m_{11}} \otimes (\alpha_1 \circ \beta_2)|_{m_{12}} \otimes \dots \otimes (\alpha_1 \circ \beta_p)|_{m_{1p}} \otimes (\alpha_2 \circ \beta_1)|_{m_{21}} \otimes \dots \otimes (\alpha_k \circ \beta_p)|_{m_{kp}} \right)$$
$$\circ\ \sigma \circ \delta_{kp}$$

(by Lemma 2.9) which is equal to :

$$conc_{pk} \circ \left(\sum_{m_{ij}} (\alpha_1 \circ \beta_1)|_{m_{11}} \otimes (\alpha_1 \circ \beta_2)|_{m_{12}} \otimes \dots \otimes (\alpha_1 \circ \beta_p)|_{m_{1p}} \otimes (\alpha_2 \circ \beta_1)|_{m_{21}} \otimes \dots \otimes (\alpha_k \circ \beta_p)|_{m_{kp}} \right) \circ \delta_{kp}$$

(by Lemma 2.10).

This concludes the proof. □

3. AN APPLICATION : A SUBALGEBRA OF $\mathbb{Q}\mathcal{B}_n$

DEFINITION 3.1. Let σ be a permutation of the symmetric group $\mathcal{S}_n$, we say that σ presents a *descent* in $i \in \{1, 2, \dots, n-1\}$ if $\sigma(i) > \sigma(i+1)$.

For any subset E of $\{1, 2, \dots, n-1\}$, we denote by $D_{\subseteq E}$ (or, when there is no doubt, simply D_E) the sum in the algebra $\mathbb{Q}\mathcal{S}_n$ of all permutations whose descent set is contained in E, we denote by $D_{=E}$ the sum in the algebra $\mathbb{Q}\mathcal{S}_n$ of all permutations whose descent set is equal to E. These elements are called *descent classes*. Each of these two families of 2^{n-1} elements spans of course the same subspace, which is indeed is a subalgebra of $\mathbb{Q}\mathcal{S}_n$, called the *Solomon descent algebra* Σ_n.

If G is a Coxeter group and W the set of its generators, then it is possible to define descents of an element $g \in G$ in $w \in W$. Of course this definition coincides with the previous one in the case $G = \mathcal{S}_n$. The paper of Solomon [So] shows that for all Coxeter group G, the subspace of $\mathbb{Q}G$ spanned by all the sums $D_{\subseteq E}$ of all elements of G whose descent set is contained in a fixed set $E \subseteq W$ is a subalgebra of $\mathbb{Q}G$.

For the hyperoctahedral group $\mathcal{B}_n$, a combinatorial equivalent definition of descents is given.

If σ is an element of the hyperoctahedral group $\mathcal{B}_n$, we represent σ as a word $\sigma_1\sigma_2\dots\sigma_n$ in the alphabet $\{\pm 1, \pm 2, \dots, \pm n\}$ such that the word $|\sigma_1||\sigma_2|\dots|\sigma_n|$ is a permutation of the symmetric group $\mathcal{S}_n$.

DEFINITION 3.2. Let σ be a permutation of the hyperoctahedral group $\mathcal{B}_n$, we say that σ presents a *descent in* $i \in \{1, 2, \dots, n-1\}$ if $\sigma(i) > \sigma(i+1)$; moreover we say that σ presents a *descent in* 0 if $\sigma(1) < 0$.

We will denote the descent algebra of $\mathcal{B}_n$ by $\Sigma\mathcal{B}_n$.

It is known (see for exemple [G-Re]) that Σ_n is anti-isomorphic to the component Γ_n of the Convolution Algebra Γ that we mentioned in Section 2, where Γ_n is the sub-algebra of Γ spanned by the set : $\{q_C \in \Gamma \mid weight(C) = n\}$. We want to show that a particular choice of the endomorphisms in our general result, leads to a subalgebra of $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle A\rangle)$ which is anti-isomorphic to a subalgebra of $\mathbb{Q}\,\mathcal{B}_n$ containing $\Sigma\mathcal{B}_n$.

Descent classes spanning the Solomon's descent algebra of the symmetric group, can be indexed by compositions of the integer n, this is easily done by using the natural bijection between compositions of n and subsets of $\{1, 2, ..., n-1\}$. This bijection associates to the composition $c_1\, c_2\, ...\, c_k$ the subset $\{\, c_1, c_1+c_2, ..., c_1+c_2+ ...+c_{k-1}\}$. We will need more general objects to describe our new classes.

DEFINITION 3.3. If n is a positive integer we call *signed composition* of n a sequence of integers $c_1\, c_2\, ...\, c_k$ such that $|c_j| > 0$ and $\sum_{j=1}^{k} |c_j| = n$.

We will usually write a signed composition of n by putting the sign '–' of the negative parts above the corresponding integer rather than on its left.

EXAMPLE. $C = \bar{1}\ 2\ 2\ \bar{2}\ \bar{3}\ 1$ is a signed composition of $1+2+2+2+3+1 = 11$.

Given a permutation $\sigma = \sigma_1\sigma_2 ...\sigma_n$ of $\mathcal{B}_n$, its *descent shape* is the signed composition of n :

$$C(\sigma) = c_1\ c_2\ ...\ c_k$$

such that, when viewed as a word, $\sigma = u_1\, u_2 ...\, u_k$ where each u_j is a word of length $|c_j|$ in the alphabet $\{1, 2, ..., n\}$, then the integers in each u_j appear in increasing order of their absolute values and they have the same sign as c_j ; we require moreover that the number k is the minimal natural number such that this decomposition is possible. We will refer to the word u_j as the *segment of σ corresponding to c_j*.

EXAMPLE. The permutation $\sigma = \bar{7}\ 5\ 8\ 3\ 11\ \bar{1}\ \bar{6}\ \bar{2}\ \bar{4}\ \bar{9}\ 10$ has descent shape $C(\sigma) = \bar{1}\ 2\ 2\ \bar{2}\ \bar{3}\ 1$, the segment of σ corresponding to $c_5 = \bar{3}$ is $\bar{2}\ \bar{4}\ \bar{9}$, because $\sigma = \bar{7}\ \ 5\ 8\ \ 3\ 11\ \ \ \bar{1}\ \bar{6}\ \ \ \bar{2}\ \bar{4}\ \bar{9}\ \ 10$.

DEFINITION 3.4. Let C and D be two signed compositions of n, we say that C is *finer* than D iff any part of D can be obtained by summing *consecutive parts of C having the same sign.*

EXAMPLE. The signed composition $\bar{1}\ 2\ 2\ \bar{2}\ \bar{3}\ 1$ is finer than the signed composition $\bar{1}\ 4\ \bar{5}\ 1$ (and the latter composition is minimal relatively to the partial order : $C \succ D$ iff C is finer than D).

DEFINITION 3.5. Let C be a signed composition of n, the *descent-shape class* (or simply *shape class*) corresponding to C is the sum in the algebra $\mathbb{Q}\mathcal{B}_n$ of all permutations whose descent shape is less fine or equal to C. We will denote this sum with $S_{\subseteq C}$, or when no doubt is possible by S_C. We will denote the sum in the algebra $\mathbb{Q}\mathcal{B}_n$ of all permutations whose descent shape is equal to C with the symbol $S_{=C}$.

Let $\Omega\mathcal{B}_n$ be the subspace of $\mathbb{Q}\mathcal{B}_n$ spanned by all the S_C.

PROPOSITION 3.6. *The dimension of* $\Omega\mathcal{B}_n$ is $2*3^{n-1}$.

Proof. It follows straightforwardly by counting the number of signed composition of the integer n. □

Here we give as example the table of the descents shapes of the elements of $\mathcal{B}_3$; for each signed composition C, the table reports the sum $S_{=C}$ of all permutations having C as descent shape.

C	$S_{=C}$	C	$S_{=C}$
3	$1\,2\,3$	$1\,1\,\bar{1}$	$2\,1\,\bar{3}+3\,1\,\bar{2}+3\,2\,\bar{1}$
$2\,1$	$1\,3\,2+2\,3\,1$	$\bar{2}\,1$	$\bar{1}\,\bar{2}\,3+\bar{1}\,\bar{3}\,2+\bar{2}\,\bar{3}\,1$
$1\,1\,1$	$3\,2\,1$	$\bar{1}\,\bar{1}\,1$	$\bar{2}\,\bar{1}\,3+\bar{3}\,\bar{1}\,2+\bar{3}\,\bar{2}\,1$
$1\,2$	$3\,1\,2+2\,1\,3$	$1\,\bar{2}$	$1\,\bar{2}\,\bar{3}+2\,\bar{1}\,\bar{3}+3\,\bar{1}\,\bar{2}$
$\bar{3}$	$\bar{1}\,\bar{2}\,\bar{3}$	$1\,\bar{1}\,\bar{1}$	$1\,\bar{3}\,\bar{2}+2\,\bar{3}\,\bar{1}+3\,\bar{2}\,\bar{1}$
$\bar{2}\,\bar{1}$	$\bar{1}\,\bar{3}\,\bar{2}+\bar{2}\,\bar{3}\,\bar{1}$	$\bar{1}\,2$	$\bar{1}\,2\,3+\bar{2}\,1\,3+\bar{3}\,1\,2$
$\overline{111}$	$\bar{3}\,\bar{2}\,\bar{1}$	$\bar{1}\,1\,1$	$\bar{1}\,3\,2+\bar{2}\,3\,1+\bar{3}\,2\,1$
$\overline{12}$	$\bar{3}\,\bar{1}\,\bar{2}+\bar{2}\,\bar{1}\,\bar{3}$	$\bar{1}\,1\,\bar{1}$	$\bar{1}\,2\,\bar{3}+\bar{2}\,1\,\bar{3}+\bar{3}\,1\,\bar{2}+\bar{1}\,3\,\bar{2}+\bar{2}\,3\,\bar{1}+\bar{3}\,2\,\bar{1}$
$2\,\bar{1}$	$1\,2\,\bar{3}+1\,3\,\bar{2}+2\,3\,\bar{1}$	$1\,\bar{1}\,1$	$1\,\bar{2}\,3+1\,\bar{3}\,2+2\,\bar{1}\,3+2\,\bar{3}\,1+3\,\bar{1}\,2+3\,\bar{2}\,1$

REMARK. The subspace $\Omega\mathcal{B}_n$ contains (an isomorphic copy of) the space Σ_n. This is clear because the descent-shape classe corresponding to an ordinary composition C of n is exactly the descent class of Σ_n associated to C.

REMARK. The subspace $\Omega\mathcal{B}_n$ contains the space $\Sigma\mathcal{B}_n$. First, it is easy to see that the elements $D_{=E}$, sums of all permutations whose descent set is exactly E, are a sum of elements $S_{=C}$, sum of all the permutation whose descent shape is exactly the signed composition C. Indeed, all the permutations in the sum $S_{=C}$ have the same descent set. Secondly, the elements $S_{=C}$ can be easily expressed as linear combination of the S_C.

To prove that $\Omega\mathcal{B}_n$ is an algebra we will use Corollary 2.3, to apply it we need to introduce some endomorphisms of a suitable free associative algebra.

Let A be a set of symbol (alphabet) and let $\bar{A}$ be the set $\{ \bar{a} \ / \ a \in A \}$. Let $\mathbb{Q}\langle A \cup \bar{A} \rangle$ be the free associative algebra over the alphabet $A \cup \bar{A}$, that is, the algebra of all polynomials in the letters of A and $\bar{A}$.

We denote by *bar* the algebra endomorphism changing the bars over the letters of the alphabet $A \cup \bar{A}$:

$$bar(a) = \bar{a} \quad \text{and} \quad bar(\bar{a}) = a \, ;$$

bar is of course degree preserving.

Again, for each n we consider the map $q_n = id|_n$ which is the identity over the space $\mathbb{Q}\langle A \cup \bar{A} \rangle_n$ of all homogeneous polynomials of degree n and which sends all the spaces $\mathbb{Q}\langle A \cup \bar{A} \rangle_m$ onto 0 if $m \neq n$. We also consider the applications $\bar{q}_n = bar|_n$ which send all the spaces $\mathbb{Q}\langle A \cup \bar{A} \rangle_m$ onto 0 if $m \neq n$, and change the bars of all the letters of a monomial of degree n.

Let $\bar{\Gamma}$ denote the convolution subalgebra of $\mathrm{End}_{\mathbb{Q}}(\mathbb{Q}\langle A \cup \bar{A} \rangle)$ generated by the q_n's and by the $\bar{q}_n$'s.

We define analogously a *pseudo signed composition* of $n \geq 0$ as a k-tuple $H = h_1 h_2 \dots h_k$ such that $\sum_{i=1}^{k} |h_j| = n$. The integer n is called the *weight* of H and denoted by $|H|$, the integer k is called *length* of H and denoted by $\ell(H)$.

To any pseudo signed composition H we associate a pseudo composition denoted by H^+ whose parts are the absolute values of the parts of H.

If $M = (m_{ij})$ is a matrix of signed integers we still denote by $C(M)$ the *pseudo signed composition associated to* M, that is :

$$C(M) = (m_{11}, m_{12}, \dots, m_{1p}, m_{21}, m_{22}, \dots, m_{2p}, \dots, m_{k1}, m_{k2}, \dots, m_{kp}) \, .$$

We have a bijection between convolution products of type

$$\alpha_1|_{h_1} * \alpha_2|_{h_2} * \dots * \alpha_p|_{h_p} \quad \text{where } \alpha_i \in \{id, bar\}$$

and signed compositions $c_1 c_2 \dots c_k$, this bijection is defined by $c_i = h_i$ if $\alpha_i = id$ and $c_i = -h_i$ otherwise. In other terms, if C is a pseudo signed composition $c_1 \ c_2 \dots c_k$, then we define the endomorphism q_C of $\mathbb{Q}\langle A \cup \bar{A} \rangle$ by :

$$q_C = q_{c_1} * q_{c_2} * \dots * q_{c_k}$$

with the convention that if $c_j < 0$ then $q_{c_j} = \overline{q_{|c_j|}}$

Our general result (Corollary 2.3.) gives the following:

COROLLARY 3.7. *The algebra $\bar\Gamma$ is closed under composition, and hence is an algebra relatively to + and ∘.*

Proof. This is a particular case of Corollary 2.3 in which the α_i's and the β_j's are all taken in the set $\{id, bar\}$ which is a set closed under composition.

Theorem 2.2 give us the multiplication table for the algebra $\bar\Gamma$.

THEOREM 3.8. *Let H, L be two pseudo signed compositions. Then :*

$$q_H \circ q_L = \sum q_{C(M)}$$

where the sum is extended to all the matrices M of integers with the two following properties :

i) the matrix M^+, obtained from M by taking the absolute values of its integers, has row sum and column sum respectively equal to H^+ and L^+,

ii) the sign of the integer M_{ij} is the product of the signs of the i-th part of H and of the j-th part of L (with the usual rule of multiplication of signs).

EXAMPLE. Let $H = 1\ \bar 3\ \bar 1$, $L = \bar 2\ 3$, then the compatible matrices are :

$$\begin{array}{c} \\ 1 \\ \bar 3 \\ \bar 1 \end{array}
\begin{array}{c} \bar 2 \quad 3 \\ \left|\begin{array}{cc} \bar 1 & 0 \\ 1 & \bar 2 \\ 0 & \bar 1 \end{array}\right| \end{array},\quad
\begin{array}{c} \bar 2 \quad 3 \\ \left|\begin{array}{cc} \bar 1 & 0 \\ 0 & \bar 3 \\ 1 & \bar 0 \end{array}\right| \end{array},\quad
\begin{array}{c} \bar 2 \quad 3 \\ \left|\begin{array}{cc} \bar 0 & 1 \\ 1 & \bar 2 \\ 1 & \bar 0 \end{array}\right| \end{array}\quad \text{and} \quad
\begin{array}{c} \bar 2 \quad 3 \\ \left|\begin{array}{cc} \bar 0 & 1 \\ 2 & \bar 1 \\ 0 & \bar 1 \end{array}\right| \end{array}.$$

Thus,

$$q_{\bar 2 3} \cdot q_{1\bar 3\bar 1} = q_{\bar 1 1 \bar 2 \bar 1} + q_{\bar 1 \bar 3 1} + q_{1 1 \bar 2 1} + q_{1 2 \bar 1 \bar 1}.$$

We shall prove now that the subspace $\Omega\mathcal{B}_n$ is anti-isomorphic to the algebra $\bar\Gamma_n$, the subalgebra of $\bar\Gamma$ spanned by the set $\{q_C \in \bar\Gamma \mid weight(C) = n\}$.

The hyperoctahedral group $\mathcal{B}_n$ acts from the right on the monomials of degree n of $\mathbb{Q}\langle A \cup \bar A\rangle$, this action can be extended to an action of the algebra $\mathbb{Q}\mathcal{B}_n$ over $\mathbb{Q}\langle A\cup\bar A\rangle_n$.

THEOREM 3.9. *The subspace $\Omega\mathcal{B}_n$ is a subalgebra of $\mathbb{Q}\mathcal{B}_n$. If $|A| \geq n$ then the linear mapping :*

$$\Omega\mathcal{B}_n \longrightarrow \bar{\Gamma}_n, \quad S_C \to q_C$$

for any signed composition C of n is an anti-isomorphism of algebras.

Proof. Suppose $C = c_1 c_2 \ldots c_k$. By linearity, it suffices to show that the action on the right of S_C is the same as the action on the left of q_C.

Let $w = a_1 a_2 \ldots a_n$ be a word of length n, then wS_C is a sum of all words $a_{\sigma(1)}a_{\sigma(2)}..a_{\sigma(n)}$ where σ is a permutation whose descent shape is less fine or equal to C. On the other side, $q_C(w)$ is the sum :

$$\sum_{u_1 \ldots u_k \in A^*} (u_1 ш\, u_2 ш \ldots ш\, u_k, w)\, q_{c_1}(u_1)\ldots q_{c_k}(u_k).$$

Since both these sums are multiplicity free, it suffices to show that a word is in the first sum iff it is in the second.

Let $a_{\sigma(1)}a_{\sigma(2)}..a_{\sigma(n)}$ be in wS_C, then factorize the word $a_{\sigma(1)}a_{\sigma(2)}..a_{\sigma(n)}$ in factors $v_1, v_2, \ldots, v_k$ of length $c_1, c_2, \ldots, c_k$, respectively.

Let $u_i = q_{c_i}(v_i)$, we shall show that w is in the shuffle : $u_1 ш\, u_2 ш \ldots ш\, u_k$.

First, we remark that the absolute value of the indices of the letters in any word u_i are in the increasing order, this is because the descent shape of σ is less fine or equal to C and we are factorizing the word $\sigma(1)\sigma(2)..\sigma(n)$ in segments of length equal to the parts of C. Secondly, we have that the letters in u_i are barred iff they are in w ; indeed, if the action of σ changed the bar of the j-th letter, then it means that the integer j belongs to a barred segment of C, let us say c_i, so its bar has been changed again by taking $q_{c_i}(v_i)$.

Then $q_{c_1}(u_1)\ldots q_{c_k}(u_k)$ is in $q_C(w)$ but,

$$q_{c_1}(u_1)\ldots q_{c_k}(u_k) = q_{c_1}(q_{c_1}(v_1))\ldots q_{c_k}(q_{c_k}(v_k)) = v_1\, v_2 \ldots v_k = a_{\sigma(1)}a_{\sigma(2)}\ldots a_{\sigma(n)}.$$

the converse can be proved analoguously. □

4. A Further Generalization

The result established in Theorem 2.2. admits a further generalization. Let us introduce the operator $\tilde{}$ which acts over a word $w = w_1 w_2 \ldots w_k$ reversing the order of its letters : $\tilde{w} = w_k \ldots w_2 w_1$. We remark that $\tilde{}$ can be extended to a linear endomorphism of $\mathbb{Q}\langle A \cup \bar{A} \rangle$.

We will denote by $\tilde{q}_n$ the restriction $\tilde{}|_n$ of the operator $\tilde{}$ to the subspace of homogeneous polynomials of degree n.

More generally, if α is a degree-preserving algebra endomorphism of $\mathbb{Q}\langle A \cup \bar{A} \rangle$, we denote by $\tilde{\alpha}|_n$ the linear map which coincides with $\tilde{q}_n \circ \alpha = \alpha \circ \tilde{q}_n$ over the subspace $\mathbb{Q}\langle A \cup \bar{A} \rangle_n$

and which is equal to zero over the spaces $\mathbb{Q}\langle A\cup\bar{A}\rangle_m$ when $m \neq n$.

We define *tilded composition* of the integer n any sequence of integers $H = h_1 h_2 \dots h_k$ whose parts are possibly tilded and such that their sum (neglecting the tildes) is equal to n.
If H is a tilded composition of n then we denote by $\|H\|$ the ordinary composition of n obtained from H by neglecting the tildes.

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be degree-preserving elements of $End_{\mathbb{Q}}\mathbb{Q}\langle A\cup\bar{A}\rangle$, if $H = h_1 h_2 \dots h_k$ is a tilded composition of the integer n we still denote by $(\alpha_1, \alpha_2, \dots, \alpha_k)_H$ the convolution product:

$$(\alpha_1|_{h_1} * \alpha_2|_{h_2} * \dots * \alpha_k|_{h_k})$$

with the convention that, if h_j is a tilded number, say $h_j = \tilde{m}$, then

$$\alpha_j|_{h_j} = \tilde{\alpha}_j|_m = \tilde{q}_m \circ \alpha_j = \alpha_j \circ \tilde{q}_m .$$

If $L = l_1 l_2 \dots l_p$ is another tilded composition of the same integer n, and $\beta_1, \beta_2, \dots, \beta_p$ other degree-preserving homomorphisms, then we will show that it is still possible to give a formula to compute :

$$(\alpha_1, \alpha_2, \dots, \alpha_k)_H \circ (\beta_1, \beta_2, \dots, \beta_p)_L$$

DEFINITION 4.1. *Let $H=h_1 h_2 \dots h_k$ and $L= l_1 l_2 \dots l_p$ be two tilded compositions of the integer n. We say that a matrix $M=(m_{ij})_{1\le i\le k,\ 1\le j\le p}$ is compatible with the two tilded compositions H and L, if M has row sum equal to $\|H\|$, column sum equal to $\|L\|$, and the number m_{ij} is tilded iff the integer h_i or (exclusive) the integer l_j are tilded.*

EXAMPLE. Let $H = \tilde{6}\ 1\ 2\ \tilde{3}$ and $L = \tilde{5}\ 3\ 4$ then,

$$\begin{array}{cccc} \tilde{5} & 3 & 4 & \\ \end{array}$$
$$M = \left|\begin{array}{ccc} 3 & \tilde{1} & \tilde{2} \\ \tilde{0} & 0 & 1 \\ \tilde{1} & 0 & 1 \\ 1 & \tilde{2} & \tilde{0} \end{array}\right| \begin{array}{c} \tilde{6} \\ 1 \\ 2 \\ \tilde{3} \end{array}$$

is compatible with H and L.

DEFINITION 4.2. *Let $p \ge 1$ and $H=h_1 h_2 \dots h_k$ be a tilded composition and $\mathcal{A} = a_1, a_2, \dots, a_p, a_{p+1}, \dots, a_{kp}$ a sequence of $k\cdot p$ elements, then the map $f_H(\mathcal{A})$ is the sequence obtained from $\mathcal{A}$ by reversing the order of the elements $a_{jp-p+1}, a_{jp-p+2}, \dots, a_{jp}$ iff the* i-th part of H is tilded.

EXAMPLE : Let $H = \bar{\bar{2}}\ 2\ \tilde{1}\ 3$, let $p = 3$ and take the sequence :

$$\mathcal{A} = a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8,\ a_9,\ a_{10},\ a_{11},\ a_{12} \quad \text{then}$$

$$f_H(\mathcal{A}) = a_3,\ a_2,\ a_1,\ a_4,\ a_5,\ a_6,\ a_9,\ a_8,\ a_7,\ a_{10},\ a_{11},\ a_{12}\,.$$

THEOREM 4.3. *Let $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_p$ be degree preserving endomorphisms of a free associative algebra $\mathbb{Q}\langle A\rangle$, let $H = h_1\ h_2 \dots h_k$ and $L = l_1\ l_2 \dots l_p$ be two tilded composition of the integer n, then the composition product :*

$$(\alpha_1, \alpha_2, \dots, \alpha_k)_H \circ (\beta_1, \beta_2, \dots, \beta_p)_L$$

is the sum of all the elements :

$$\Big(f_H((\alpha_1 \circ \beta_1), (\alpha_1 \circ \beta_2), \dots, (\alpha_1 \circ \beta_p), \\ (\alpha_2 \circ \beta_1), (\alpha_2 \circ \beta_2), \dots, (\alpha_2 \circ \beta_p), \\ \dots \\ (\alpha_k \circ \beta_1), (\alpha_k \circ \beta_2), \dots, (\alpha_k \circ \beta_p))\Big)_{f_H(C(M))}$$

such that the matrix M is compatible with H and L.

To prove this theorem, we need only few lemmas, since most of results hold trivially true when we consider tilded compositions instead of ordinary ones. We state these lemmas without giving the proof, which is easy but full of calculations.

LEMMA 4.4. *One has the formula*

$$\delta_p \circ \tilde{\alpha}|_n = \sum_{h_1 + \dots + h_p = n} (\tilde{\alpha}|_{h_1} \otimes \tilde{\alpha}|_{h_2} \otimes \dots \otimes \tilde{\alpha}|_{h_p}) \circ \delta_p . \square$$

We also have the analogous of Lemma 2.6 when endomorphisms are possibly composed with the $\bar{q}_n$'s.

LEMMA 4.5. *We have for two tilded compositions $H = h_1\ h_2 \dots h_p$ and $L = l_1\ l_2 \dots l_p$ and for algebra endomorphisms $\alpha_1, \alpha_2, \dots, \alpha_p, \beta_1, \beta_2, \dots, \beta_p$:*

$$(\alpha_1, \alpha_2, \dots, \alpha_p)_{\otimes H} \circ (\beta_1, \beta_2, \dots, \beta_p)_{\otimes L} = \begin{cases} (\alpha_1 \circ \beta_1,\ \alpha_2 \circ \beta_2,\ \dots,\ \alpha_p \circ \beta_p)_{H \cdot L} & \text{if } \|H\| = \|L\| \\ 0 & \text{otherwise} \end{cases}$$

where $H \sim L$ is the tilded composition such that $\|H \sim L\| = \|H\| = \|L\|$ and its i-th part is tilded iff the i-th part of H or (exclusive) the i-th part of L are tilded. □

Proof of Theorem 4.3. The reader can easily obtain the proof just following the proof of Theorem 2.2 but using Lemmas 4.4 and 4.5 and considering that :

$$\tilde{q}_l \circ conc_{i_1,\dots,i_k} = conc_{i_k,\dots,i_1} \circ \sum_{m_1+\dots+m_k=l} \tilde{q}_{m_1} \otimes \dots \otimes \tilde{q}_{m_k}. \square$$

In other terms, the rule is similar to the one in Theorem 2.2 but composing tildes as elements of order two, and reading the rows of the matrix corresponding to the tilded parts of H from right to left. An example will make this clearer.

EXAMPLE. Let $H = 2\ 5\ \tilde{4}$ and $L = \tilde{4}\ \tilde{3}\ 4$. In the decomposition of

$$(\alpha_1, \alpha_2, \alpha_3)_H \circ (\beta_1, \beta_2, \beta_3)_L$$

one of the terms of the resulting sum will be :

$$(\widetilde{\alpha_1 \circ \beta_1})|_1 \otimes (\widetilde{\alpha_1 \circ \beta_2})|_0 \otimes (\alpha_1 \circ \beta_3)|_1 \otimes$$

$$(\widetilde{\alpha_2 \circ \beta_1})|_1 \otimes (\widetilde{\alpha_2 \circ \beta_2})|_2 \otimes (\alpha_2 \circ \beta_3)|_2 \otimes$$

$$(\widetilde{\alpha_3 \circ \beta_3})|_1 \otimes (\alpha_3 \circ \beta_2)|_1 \otimes (\alpha_3 \circ \beta_1)|_2 .$$

because of the following matrix M which is compatible with H and L :

H \ L	$\tilde{4}$	$\tilde{3}$	4
2	$\tilde{1}$	$\tilde{0}$	1
5	$\tilde{1}$	$\tilde{2}$	2
$\tilde{4}$	2	1	$\tilde{1}$

5. ANOTHER APPLICATION : THE MULTIPLICATION RULE IN THE DESCENT ALGEBRA OF $\mathcal{B}_N$

If $E = \{s_0, s_1, \dots, s_{k-1}\}$ is a subset of $\{0, 1, \dots, n-1\}$ then B_E denotes the descent class of all permutations of $\mathcal{B}_n$ whose descent set is included in E ; we will rather index this descent class by the composition $C = c_1 c_2 \ldots c_k$ of the integer $m = n - s_0$ whose parts are defined by :

$$c_j = s_j - s_{j-1} \text{ if } 0 < j < k \quad \text{and} \quad c_k = n - s_{k-1} .$$

since such composition is in bijection with E.

EXAMPLE : If $E = \{2, 5, 6, 9, 10\} \subset \{0, 1, \ldots, 10\}$ (here $n=11$) then $C = 3\ 1\ 3\ 1\ 1$.

We have already pointed out that all descent classes of $\mathfrak{B}_n$ are in the algebra $\Omega\mathfrak{B}_n$ and hence they are a linear combination of shape classes ; we can give the explicit expression of a descent class in function of some other elements of $\Omega\mathfrak{B}_n$ strictly related to shape classes.

DEFINITION 5.1. *If $H = h_1 h_2 \ldots h_k$ is a signed composition of n and S_H is the shape class associated to H, then we denote by $\tilde{S}_H$ the sum of those permutations of $\mathfrak{B}_n$ obtained from the permutations of S_H by inverting the order of the segments corresponding to the negative parts of the composition H.*

EXAMPLE. In $\Omega\mathfrak{B}_4$, take the shape class :

$$\begin{aligned} S_{\bar{2}\bar{1}1} = {} & \bar{1}\bar{2}\bar{3}4 + \bar{1}\bar{2}\bar{4}3 + \bar{2}\bar{3}\bar{4}1 + \bar{1}\bar{3}\bar{4}2 + \bar{1}\bar{3}\bar{2}4 + \bar{1}\bar{4}\bar{2}3 + \\ & \bar{2}\bar{4}\bar{3}1 + \bar{1}\bar{4}\bar{3}2 + \bar{2}\bar{3}\bar{1}4 + \bar{2}\bar{4}\bar{1}3 + \bar{3}\bar{4}\bar{2}1 + \bar{3}\bar{4}\bar{1}2, \end{aligned}$$

then $\tilde{S}_{\bar{2}\bar{1}1}$ is the sum :

$$\begin{aligned} & \bar{2}\bar{1}\bar{3}4 + \bar{2}\bar{1}\bar{4}3 + \bar{3}\bar{2}\bar{4}1 + \bar{3}\bar{1}\bar{4}2 + \bar{3}\bar{1}\bar{2}4 + \bar{4}\bar{1}\bar{2}3 + \\ & \bar{4}\bar{2}\bar{3}1 + \bar{4}\bar{1}\bar{3}2 + \bar{3}\bar{2}\bar{1}4 + \bar{4}\bar{2}\bar{1}3 + \bar{4}\bar{3}\bar{2}1 + \bar{4}\bar{3}\bar{1}2. \end{aligned}$$

THEOREM 5.2. *Let $E = \{s_0, s_1, \ldots, s_{k-1}\}$ be a subset of $\{0, 1, \ldots, n-1\}$ and $C = c_1 c_2 \ldots c_k$ the corresponding composition of $m = n - s_0$. If B_C denotes the corresponding descent class of $\mathfrak{B}_n$, then we have the expression :*

$$B_C = \sum_H \tilde{S}_H .$$

where the sum is extended to all signed compositions $H = h_0 \bar{h}_1 h'_1 \bar{h}_2 h'_2 \ldots \bar{h}_k h'_k$ of n such that $h_0 = s_0$ and $h_i + h'_i = c_i$ for all i between 1 and k.

Proof. A permutation σ is in the sum $\sum_H \tilde{S}_H$ iff there exists a signed composition :

$$M = m_0 \bar{m}_1 m'_1 \bar{m}_2 m'_2 \ldots \bar{m}_k m'_k$$

such that σ is in the sum $\tilde{S}_M$ and the set :

$$\{m_0,\ m_0+m_1+m'_1,\ m_0+m_1+m'_1+m_2+m'_2,\ \ldots,\ m_0+m_1+m'_1+m_2+m'_2\ +\ldots+m_{k-1}+m'_{k-1}\}$$

is equal to

$$\{s_0,\ s_0+c_1,\ s_0+c_1+c_2,\ \ldots,\ s_0+c_1+c_2+\ \ldots+c_{k-1}\} = E.$$

Now, σ is in the sum $\tilde{S}_M$ iff when we factorize it in segments of length equal to the parts of M, then in each of the segments the integers appear in the increasing order. This is because in a permutation of S_M the integers in the segments corresponding to the different parts of the composition M appear in increasing order of their absolute values and in $\tilde{S}_M$ the order of the barred segments is inverted.

This assertion is equivalent to say that there is no descent inside the segments corresponding to the parts of M ; of course, neither there are at the end of the barred segments, so the descents of σ can be only at the end of the positive segments, and hence :

$Des(\sigma) \subseteq$

$$\subseteq \{m_0,\ m_0+m_1+m'_1,\ m_0+m_1+m'_1+m_2+m'_2,\ \ldots,\ m_0+m_1+m'_1+m_2+m'_2+\ \ldots+m_{k-1}+m'_{k-1}\} = E$$

That is, σ is in B_C. □

If we know the way to multiply the elements $\tilde{S}_H$, we can deduce the multiplication rule for descent classes of $\mathcal{B}_n$ which has been presented in [B-B]. Similarly to the action of the S_H's, the action of the sums $\tilde{S}_H$ over the elements of the free associative algebra $\mathbb{Q}\langle A\cup\bar{A}\rangle$ corresponds to the action of suitable convolution products of endomorphisms of $\mathbb{Q}\langle A\cup\bar{A}\rangle$.

Theorem 4.3, if applied to the free associative algebra $\mathbb{Q}\langle A\cup\bar{A}\rangle$, gives us the rule to multiply convolution products in which there appear not only the q_i's and the $\bar{q}_i$'s, but also the $\tilde{q}_i$'s and the $\tilde{\bar{q}}_i$'s. These elements can be naturally indexed by compositions whose parts can possibly be overbarred <u>and</u> tilded. We will refer to these objects as *tilded signed compositions.*

If H is a tilded signed composition we still indicate with H^+ the (ordinary) composition obtained from H by neglecting the signs and the tildes.

We formalize this rule in the following (consequence of Theorem 4.3):

COROLLARY 5.3. *Let H and L be two tilded signed compositions of the same integer n, then :*

$$q_H * q_L = \sum q_{f_H C(M)}$$

where the sum is extended to all the tilded signed matrices M compatible with the tilded signed compositions H and L and such that the sign of the integer $M_{i,j}$ is the product of the signs of the i-th part of H and the j-th part of L.

In other words, to compute $q_H \circ q_L$, one has to fill a $\ell(H)\times\ell(L)$ matrix with integers in all the possible ways such that :

i) the row sum of M is H^+ and the column sum of M is L^+,

ii) the sign of the integer $M_{i,j}$ is the product of the signs of the i-th part of H and the j-th part of L,

iii) the integer $M_{i,j}$ is tilded iff the i-th part of H or (exclusive) the j-th part of L are tilded.

iv) the lines of the matrix corresponding to the tilded parts of H must be read from right to left.

EXAMPLE. Take $H = \bar{2}\,\tilde{\bar{3}}$ and $L = \tilde{1}\ 2\ \bar{2}$, then the possible matrices having row sum H^+ and column sum L^+ with the property *ii)* and *iii)* are :

$$\begin{vmatrix} \tilde{\bar{1}} & \bar{0} & 1 \\ \bar{0} & \tilde{\bar{2}} & \tilde{1} \end{vmatrix},\quad \begin{vmatrix} \tilde{\bar{1}} & \bar{1} & 0 \\ \bar{0} & \tilde{\bar{1}} & \tilde{2} \end{vmatrix},\quad \begin{vmatrix} \tilde{\bar{0}} & \bar{1} & 1 \\ \bar{1} & \tilde{\bar{1}} & \tilde{1} \end{vmatrix},\quad \begin{vmatrix} \tilde{\bar{0}} & \bar{2} & 0 \\ \bar{1} & \tilde{\bar{0}} & \tilde{2} \end{vmatrix} \text{ and } \begin{vmatrix} \tilde{\bar{0}} & \bar{0} & 2 \\ \bar{1} & \tilde{\bar{2}} & \tilde{0} \end{vmatrix},$$

by reading from right to left the second line of each matrix (corresponding to the tilded part $\tilde{\bar{3}}$) and neglecting the zeroes we have :

$$q_{\bar{2}\tilde{\bar{3}}} \cdot q_{\tilde{1}2\bar{2}} = q_{\tilde{\bar{1}}1\tilde{1}\tilde{\bar{2}}} + q_{\tilde{\bar{1}}\bar{1}\tilde{2}\tilde{\bar{1}}} + q_{\bar{1}1\tilde{1}\tilde{\bar{1}}\bar{1}} + q_{\bar{2}\tilde{2}\bar{1}} + q_{2\tilde{\bar{2}}\bar{1}}.$$

We leave to the reader the proof of the following proposition which gives the interpretation of the action of the elements $\tilde{S}_H$ in terms of convolution products.

PROPOSITION 5.4. *Let $H = h_1 h_2 \ldots h_k$ be a signed composition and let $\bar{S}_H$ the sum as in Definition 5.1, then the action (from the right) of $\bar{S}_H$ over the elements of $\mathbb{Q}\langle A\cup\bar{A}\rangle$ is the same as the action (from the left) of $q_{\tilde{H}}$, where $\tilde{H}$ is the composition obtained from H by putting tildes over all its negative parts.*

We are now ready to express the action of the descent classes of $\mathfrak{B}_n$ in terms of endomorphisms of the free associative algebra and using Theorem 4.3, give the rule of multiplication.

PROPOSITION 5.5. *Let* $E = \{s_0, s_1, \ldots, s_{k-1}\}$ *be a subset of* $\{0, 1, \ldots, n-1\}$, $C = c_1 c_2 \ldots c_k$ *the corresponding composition of* $m = n - s_0$, *and* B_C *the corresponding descent class of* $\mathfrak{D}_n$, *then the action from the right of the elements* B_C *is the same as the action from the left of:*

$$Q_C = \sum_{c_i' + c_i'' = c_i} \left[q_{c_0} * \prod^{*} (\bar{\bar{q}}_{c_i'} * q_{c_i''}) \right] \quad \text{where } c_0 = s_0.$$

Proof. By Theorem 5.2, $B_C = \sum_H \tilde{S}_H$, where the sum is extended to all the signed compositions $H = h_0 \bar{h}_1 h'_1 \bar{h}_2 h'_2 \ldots \bar{h}_k h'_k$ of n such that $h_0 = n - m = s_0$ and $h_i + h'_i = c_i$ for all i between 1 and k. Then by Proposition 5.4, the action of B_C is the same as that of $\sum_H \tilde{q}_H$; if we replace $\tilde{q}_H$ by its explicit expression we obtain the equality in the claim. □

Let $F = \{t_0, t_1, \ldots, t_{l-1}\}$ be another subset of $\{0, 1, \ldots, n-1\}$, $D = d_1 d_2 \ldots d_l$ the composition of $p = n - t_0$ corresponding to the subset F ; let

$$\sum_{d_j' + d_j'' = d_j} \left[q_{d_0} * \prod^{*} (\bar{\bar{q}}_{d_j'} * q_{d_j''}) \right] \qquad (\text{where } d_0 = t_0)$$

be the convolution product corresponding to the descent class B_D.

To compute $B_D B_C$(product in the algebra $\mathbb{Q}\,\mathfrak{D}_n$) it suffices to compute :

$$Q_C \circ Q_D = \sum_{c_i' + c_i'' = c_i} \left[q_{c_0} * \prod^{*} (\bar{\bar{q}}_{c_i'} * q_{c_i''}) \right] \circ \sum_{d_j' + d_j'' = d_j} \left[q_{d_0} * \prod^{*} (\bar{\bar{q}}_{d_j'} * q_{d_j''}) \right] \quad (5.1)$$

With some remarks we will show that it is not necessary to compute one by one all the compositions between any term of the first sum and any term of the second.

REMARK 5.6. The result of (5.1) is still a sum of elements of the kind: $q_{c_0} * \prod^{*} (\bar{\bar{q}}_{c_i'} * q_{c_i''})$. This follows from the rule of composition : $= \circ = \, = id$, and from the zigzaging way to read the matrices.

This will be clearer with the figure 5.1 in which we represented the type of matrices obtained when calculating the composition $Q_C \circ Q_D$ where $C = c_1 c_2 c_3 c_4$ is a composition of $n - c_0$ and $D = d_1 d_2 d_3$ a composition of $n - d_0$.

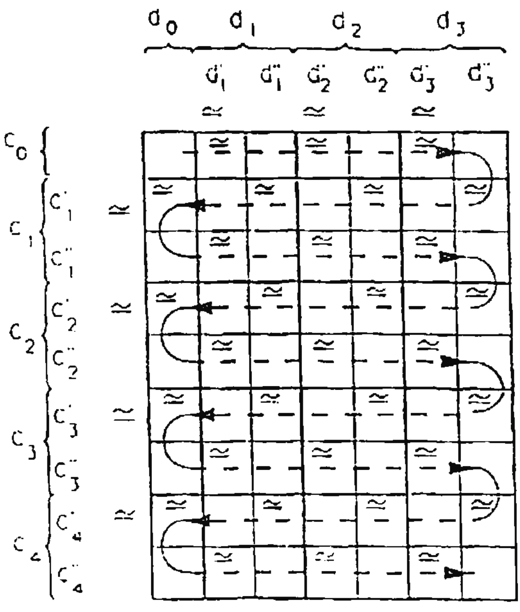


FIG. 5.1

REMARK 5.7. The result of (5.1) is still a sum of elements of the kind :

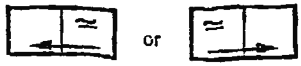
$$\sum_{h_i+h'_i=c_i} \left[q_{h_0} * \prod^{*} (\tilde{\tilde{q}}_{h_i}) * q_{h'_i}) \right] .$$

Proof. To prove this, we show that if in the decomposition of (5.1) there appears $\tilde{q}_H$ for $H = h_0 \bar{h}_1 h'_1 \bar{h}_2 h'_2 \ldots \bar{h}_p h'_p$, then there appears also any $\bar{q}_L$, with $L = l_0 \bar{l}_1 l'_1 \bar{l}_2 l'_2 \ldots \bar{l}_p l'_p$ such that, $h_0=l_0$ and $\bar{h}_i + h'_i = \bar{l}_i + l'_i$ for all i between 1 and p.
We can suppose that L is equal to $h_0 \bar{h}_1 h'_1 \ldots \bar{h}_j + a \; h'_j - a \ldots \bar{h}_p \; h'_p$ for an integer a, and then use induction.

Suppose we find $\tilde{q}_H$ when multiplying

$$\left[q_{c_0} * \tilde{\tilde{q}}_{c_1'} * q_{c_1''} * \tilde{\tilde{q}}_{c_2'} * q_{c_2''} * \ldots * \tilde{\tilde{q}}_{c_k'} * q_{c_k''} \right] \circ \left[q_{d_0} * \tilde{\tilde{q}}_{d_1'} * q_{d_1''} * \tilde{\tilde{q}}_{d_2'} * q_{d_2''} * \ldots * \tilde{\tilde{q}}_{d_l'} * q_{d_l''} \right]$$

If $\bar{h}_p$ and h'_p are in a domino of the form :



which is placed in the $2m$-th and in the $(2m+1)$-th columns, then we find $\tilde{q}_L$ when multiplying :

$$\left[q_{c_0} * \overset{\approx}{q}_{c_1'} * q_{c_1''} * \overset{\approx}{q}_{c_2'} * q_{c_2''} * \dots * \overset{\approx}{q}_{c_k'} * q_{c_k''}\right] \circ \left[q_{d_0} * \overset{\approx}{q}_{d_1'} * q_{d_1''} * \dots * \overset{\approx}{q}_{d_m'-a} * q_{d_m''+a} * \dots * \overset{\approx}{q}_{d_l'} * q_{d_l''}\right]$$

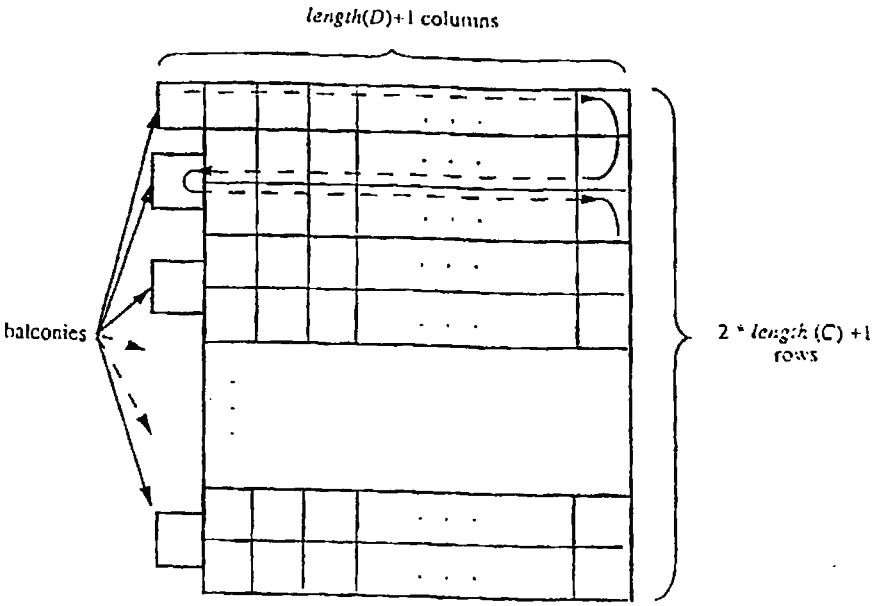
If $\bar{h}_p$ and h_p' are in a domino of the matrix like :

which is placed in the $2m$-th and in the $(2m+1)$-th rows, then we find $\tilde{q}_L$ when multiplying :

$$\left[q_{c_0} * \overset{\approx}{q}_{c_1'} * q_{c_1''} * \dots * \overset{\approx}{q}_{c_m'-a} * q_{c_m''+a} * \dots * \overset{\approx}{q}_{c_k'} * q_{c_k''}\right] \circ \left[q_{d_0} * \overset{\approx}{q}_{d_1'} * q_{d_1''} * \overset{\approx}{q}_{d_2'} * q_{d_2''} * \dots * \overset{\approx}{q}_{d_l'} * q_{d_l''}\right].$$

□

These two remarks show that the information in these dominoes is redundant if we are interested in the decomposition of (5.1) in terms of Q_C's and not of $\tilde{q}_H$'s. We can contract all such dominoes in one box and obtain the following objects that we call *templates* (as called in [B-B]), we will call *balconies* the boxes of the first column of the template.



When a template M is filled with integers it is possible to obtain a composition $C(M)$ by reading the template in the sense of the arrows. We thus have proved the result of Bergeron and Bergeron [B-B].

THEOREM 5.8. *Let* $E = \{s_0, s_1, \ldots, s_{k-1}\}$ *and* $F = \{t_0, t_1, \ldots, t_{l-1}\}$ *be two subsets of* $\{0, 1, \ldots, n-1\}$, *let* $C = c_1 c_2 \ldots c_k$ *be the composition of* $n - s_0$ *corresponding to the subset* E *and* $D = d_1 d_2 \ldots d_l$ *the composition of* $n - t_0$ *corresponding to the subset* F. *Then* :

$$B_D B_C = \sum B_{C(M)}$$

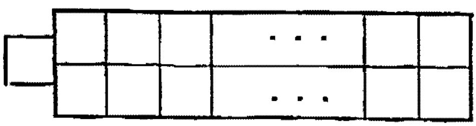
Where the sum is extended to all the templates M with the following properties :

i) the sum of the integers of the balconies is t_0.

ii) the sum of the integers of the first row is s_0.

iii) the sum of the integer of the i-th column is d_i.

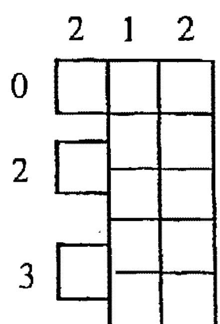
iv) the sum of the integers of the i-th block (counting from the top) of shape :



is c_i.

EXAMPLE. Let $E=\{0, 2\}$, $F=\{2, 3\}$ be subsets of $\{0, 1, 2, 3, 4\}$ (here $n=5$). Then the composition of $n - e_0 = 5 - 0 = 5$ corresponding to E is $C = 2\ 3$ and the composition of $n - f_0 = 5 - 2 = 3$ corresponding to F is $D = 1\ 2$, (see definition at the beginning of Section 5), hence we have $e_0 = 0$ and $d_0 = 2$.

To compute the product of B_C and B_D in the hyperoctahedral group $\mathfrak{B}_5$ one has to fill the following template in all the possible ways such that the row sum and the column sum are the compositions indicated.



Here is a complete list of the templates complatible with the two compositions C and D :

000	000	000	000	000	000
00	00	00	00	00	00
2 00	2 00	2 00	2 00	2 00	2 00
12	10	02	00	11	01
0 00	0 02	0 10	0 12	0 01	0 11

000	000	000	000	000	000	000	000
10	10	11	11	01	01	00	00
0 01	0 01	0 00	0 00	0 10	0 10	0 11	0 11
01	00	01	00	01	00	01	00
2 00	2 01	2 00	2 01	2 00	2 01	2 00	2 01

000	000	000	000	000	000
00	02	01	00	02	01
0 02	0 00	0 01	0 02	0 00	0 01
10	10	10	00	00	00
2 00	2 00	2 00	2 10	2 10	2 10

000	000	000	000	000	000
10	10	10	00	00	00
1 00	1 00	1 00	1 10	1 10	1 10
02	00	01	02	00	01
1 00	1 02	1 01	1 00	1 02	1 01

000	000	000	000	000	000	000	000
01	01	00	00	01	01	00	00
1 00	1 00	1 01	1 01	1 00	1 00	1 01	1 01
11	10	10	11	01	00	00	01
1 00	1 01	1 01	1 00	1 10	1 11	1 11	1 10

this gives the decomposition :

$$B_{23} * B_{12} = 4B_{221} + 4B_{212} + 2B_{2111} + 7B_{1112} + 7B_{1121} + 10B_{11111}.$$

6. Wreath Products

The hyperoctahedral group $\mathcal{B}_n$ being the wreath product of the groups $\mathcal{S}_n$ and $\mathcal{C}_2$ (the cyclic group of order 2), our purpose in this paragraph is to generalize the construction of the algebra $\Omega\mathcal{B}_n$ to the case of wreath products $\mathcal{S}_n[\mathcal{C}_p]$ for any positive integer $p>2$.

Definition 6.1. A *p-signed integer* is a couple (i, k) where i is a positive integer and k an integer modulo p.

In the examples we will represent the p-signed integer (i, k) with the number i with k bars above it.

EXAMPLE. We represent the p-signed integer $(7, 3)$ (where $p > 3$) as $\bar{\bar{\bar{7}}}$.

DEFINITION 6.2. If (i, k) is a p-signed integer, the positive integer i is called *absolute value* of (i, k), denoted by $|(i, k)|$, the positive integer k is called *sign* of (i, k), denoted by $sgn(i, k)$.

DEFINITION 6.3. If n is a positive integer, we call *p-signed composition of n* a sequence of p-signed integers $c_1\, c_2 \ldots c_k$ such that $\sum_{j=1}^{k} |c_j| = n$.

EXAMPLE. $C = \bar{1}\ 2\ 2\ \bar{\bar{\bar{2}}}\ \bar{3}\ \bar{\bar{1}}$ is a p-signed composition of $1+2+2+2+3+1 = 11$. (Here p must be greater than 3).

Let σ be an element of the wreath product $\mathcal{S}_n[\mathcal{C}_p]$, we recall that σ can be represented as a word $\sigma_1 \sigma_2 \ldots \sigma_n$ in the alphabet {*p-signed integer whose absolute value is less or equal to n*}, such that the word $|\sigma| = |\sigma_1| |\sigma_2| \ldots |\sigma_n|$ is a permutation of the symmetric group $\mathcal{S}_n$.

The product between two elements of σ and τ of $\mathcal{S}_n[\mathcal{C}_p]$ is such that $|\sigma\tau| = |\sigma||\tau|$ (usual product of permutations in $\mathcal{S}_n$) and $sgn(\sigma\tau_i) = sgn(\tau_i) + sgn(\sigma_{|\tau_i|})$ where the sum is made modulo p.

Given a permutation σ of $\mathcal{S}_n[\mathcal{C}_p]$, its *descent shape* is the p-signed composition of n :

$$C(\sigma) = c_1\, c_2 \ldots c_k$$

such that, when σ is factorized as $\sigma = u_1\, u_2 \ldots u_k$ where each u_j is a word of length $|c_j|$ in the alphabet A, then the p-signed integers in each u_j appear in increasing order of their absolute values and they all have the same sign as c_j ; we require moreover that the number k is the minimal natural number such that this decomposition is possible. We will refer to the word u_j as *the segment of σ corresponding to c_j*

EXEMPLE. The permutation $\sigma = \bar{\bar{\bar{7}}}\ 5\ 8\ 3\ 11\ \bar{1}\ \bar{6}\ \bar{2}\ \bar{4}\ \bar{9}\ \bar{\bar{10}}$ has descent shape $C(\sigma) = \bar{\bar{\bar{1}}}\, 2\, 2\, \bar{2}\, \bar{3}\, \bar{\bar{1}}$.

DEFINITION 6.4. Let C and D be two p-signed compositions of n, we say that C is *finer* than D iff any part of D is obtained by summing consecutive parts of C having the same sign.

EXAMPLE. The signed composition $\bar{\bar{\bar{1}}}\ 2\ 2\ \bar{2}\ \bar{3}\ \bar{1}$ is finer than the signed composition $\bar{\bar{1}}\ 4\ \bar{5}\ \bar{\bar{1}}$ (and no signed composition is less fine than the latter).

DEFINITION 6.5. Let C be a p-signed composition of n, the *descent-shape class* corresponding to C is the sum in the algebra $\mathbb{Q}(\mathcal{S}_n[\mathcal{C}_p])$ of all permutations whose descent shape is less fine or equal to C. We will denote this sum with S_C.

Let $\Omega(S_n[C_p])$ be the subspace of $\mathbb{Q}(\mathcal{S}_n[\mathcal{C}_p])$ spanned by all the S_C.

PROPOSITION 6.6. *The dimension of* $\Omega(\mathcal{S}_n[\mathcal{C}_p])$ *is* $p*(p+1)^{n-1}$.

Proof. Any (ordinary) composition C of the integer n with k parts gives birth to p^k different p-signed compositions of n, by giving any of the possible p signs to any of the k parts of C. Since the composition of the integer n with k parts are in number of $\binom{n-1}{k-1}$ we obtain :

$$\sum_{k=1}^{n} \binom{n-1}{k-1} p^k = p \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} = p\,(1+p)^{n-1} \qquad \square$$

The proof that $\Omega(\mathcal{S}_n[\mathcal{C}_p])$ is a subalgebra of $\mathbb{Q}(\mathcal{S}_n[\mathcal{C}_p])$ is easily deduced from our general result, Corollary 2.3, as shown in the following.

Let A be an alphabet and p a positive number. Consider the sets :

$$\bar{A}^k = \{\ \bar{a}\ /\ a \in \bar{A}^{k-1}\ \}$$ for all k between 0 and $p-1$, with the convention that $\bar{A}^0 = A$.

Let $A = \bigcup_{k=0}^{p-1} \bar{A}^k$ and $\mathbb{Q}\langle A\rangle$ the free associative algebra over the alphabet A, that is, the algebra of all polynomials in the letters of the $\bar{A}^k$'s for all k.

We denote by bar^k the algebra endomorphism which adds k bars modulo p above each letter of A ; bar^k is of course degree-preserving.

Consider for each n the linear applications $q_{k,n} = bar^k|_n$ which send all the spaces $\mathbb{Q}\langle A\rangle_m$ onto 0 if $m \neq n$, and adds k bars modulo p above the letters of a monomial of degree n.

Let $\bar{\Gamma}^p$ denote the convolution algebra of End($\mathbb{Q}\langle A\rangle$) generated by the $q_{k,n}$'s.

In an analogous way, if H is a pseudo p-signed composition $c_1\ c_2\ \ldots c_k$, then we define the endomorphism q_H of $\mathbb{Q}\langle A\rangle$ by :

$$q_H = q_{c_1} * q_{c_2} * \dots * q_{c_k}$$

with the convention that $q_{c_j} = q_{sgn(c_j),|c_j|}$.

Our general results give us the following:

COROLLARY 6.7. *The algebra* $\overline{\Gamma}^p$ *is closed under composition.*

Proof. It is a consequence of Corollary 2.3 (whose this is another particular case) and of the fact that the composition of two generators of $\overline{\Gamma}^p$ is still a sum of elements of the same form.

The same theorem give us the multiplication table for the algebra $\overline{\Gamma}^p$.

COROLLARY 6.8. *Let H, L be two pseudo p-signed compositions. Then :*

$$q_H * q_L = \sum q_{C(M)}$$

where the sum is extended to all the matrices M with the two following properties :

i) the matrix M^+ *obtained from M by taking the absolute values of its integers has row sum and the column sum respectively equal to* H^+ *and* L^+

ii) the sign of the entry M_{ij} *is the sum (modulo p) of the signs of the i-th part of H and of the j-th part of L.*

To deduce now that $\Omega(\mathcal{S}_n[\mathcal{C}_p])$ is an algebra, it suffices to prove the following theorem (analogue to Theorem 3.9).

THEOREM 6.9. *The subspace* $\Omega(\mathcal{S}_n[\mathcal{C}_p])$ *is a subalgebra of* $\mathbb{Q}(\mathcal{S}_n[\mathcal{C}_p])$. *If* $|A| \geq n$ *then the linear mapping :*

$$\Omega(\mathcal{S}_n[\mathcal{C}_p]) \longrightarrow \overline{\Gamma}^p_n, \quad S_C \to q_C$$

for any p-signed composition C of n is an anti-isomorphism of algebras.

(here, $\overline{\Gamma}^p_n$ denotes the subalgebra of $\overline{\Gamma}^p$ spanned by the set $\{q_C \in \overline{\Gamma}^p \mid weight(C) = n\}$.

We conclude with the following

REMARK. As independently observed also by P. Weil, the previous construction is easily extended to all wreath products $\mathcal{S}_n[G]$, where G is any abelian group, using the action of this group on the elements (i, g) where $i \in \{1, 2, \dots, n\}$ and $g \in G$.

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