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Palindromization and construction of Markoff triples

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ABSTRACT

The Markoff equation is the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$. A solution is called a Markoff triple. We give a bijection between the free monoid on two letters and the set of Markoff triples, using the palindromization map of Aldo de Luca. In our construction, special Christoffel words appear, whose lengths are Markoff numbers; we study their standard and palindromic factorizations, and show that they are self-dual.

1. Introduction

Among the many jewels of Combinatorics on Words discovered by Aldo de Luca, the palindromization mappings are particularly notable. They explain many phenomena in the theory of Sturmian sequences.

He defines first the *palindromic closure* $w^{(+)}$ of a word w: it is the shortest palindrome of which w is a prefix. Then he defines the *iterated palindromic closure* $Pal(w)$ of w: it maps the empty word onto itself and recursively, for any letter a, $Pal(wa) = (Pal(w)a)^{(+)}$; see [8], where Pal is denoted ψ.

He shows that this mapping, appropriately extended to infinite words, is a bijection from the free monoid on two letters onto the set of characteristic Sturmian sequences, [8] Theorem 5. Concerning finite words, Pal is a bijection from this free monoid onto the set of the so-called *central words*, [8] Proposition 8.

Central words are special palindromes and have several characterizations. As shown by Aldo de Luca and Filippo Mignosi [10], they are extremal words in regard to the Fine and Wilf periodicity theorem: they are the words having two relatively prime periods, of length the sum of these periods minus two.

Furthermore, and this is more closely related to the present article, they are words which are the proper maximal central factors of Christoffel words; the latter are the words obtained by discretizing from below a segment in the plane. For more on central words, Christoffel words and finite Sturmian words, see [14].

Another palindromization-like mapping was introduced by Aldo de Luca and Alessandro De Luca [9]. They consider *antipalindromes*, which are words on a two-letter alphabet, such that their reversal is obtained by the exchange of the two letters. The function $Antipal$ is then defined, similarly to Pal. One of their beautiful result is that $Antipal = \theta \circ Pal$, where θ is the famous Thue-Morse morphism, sending a onto ab and b onto ba.

Recently [15], Laurent Vuillon and the third author showed that the Markoff numbers (which are the components of Markoff triples, see below) could be constructed by double palindromization. Precisely: each Markoff number is the length, plus two, of the central word equal to $Pal \circ Antipal(w)$, for some binary word w.

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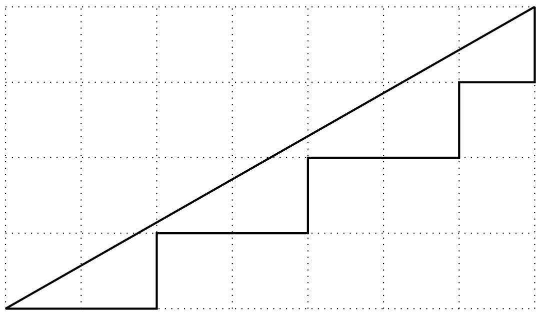

Fig. 1. The Christoffel path of slope 4/7.

The aim of the present article is to construct similarly each *Markoff triple*, that is, each solution of the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

The construction goes as follows: one constructs first the (infinite binary complete) tree of the free monoid $\{a, b\}^*$, in other words the Cayley graph of this monoid. Each vertex w is then replaced by a triple of words (the function $Trio(w)$ of Section 3) naturally associated to w. Then one replaces each element u of this triple by $2 + |Pal \circ Antipal(au)|$. The new vertices are exactly all Markoff triples, each appearing exactly once.

One motivation for this result is that one deduces immediately from the infinite complete binary tree of the free monoid on two letters a similar tree for the Markoff triples. It is well-known that Markoff triples may be written on such a tree, by the process of neighboring triples (see e.g. [1] p. 46-47). Our result gives another proof of this result. It is remarkable that this is an application of the palindromization map of Aldo de Luca.

The proof rests on a result obtained by the work of Harvey Cohn [7], Enrico Bombieri [3] and the third author [12]: a matrix bijection from the set of Christoffel words onto the set of Markoff triples, see Theorem 3.1.1 in [14].

In the final section, we give some properties of the special Christoffel words of the form $a(Pal \circ Antipal(u))b$ which appear in this construction. First, we find the standard and palindromic factorizations of these words. Recall that each Christoffel word is uniquely the product of two Christoffel words (Jean-Pierre Borel and François Laubie [4]), and also uniquely the product of two palindromes (Wai-Fong Chuan [6]); an equivalent property, for standard words (which are special conjugates of Christoffel words), was also proved by Aldo de Luca [8].

Secondly, we show that these Christoffel words are self-dual for a duality, similar, but different, of the duality studied by Valérie Berthé, Aldo de Luca and the third author [2]. This leads us again to Aldo de Luca, who *called harmonic words* the self-dual words, in an article with Arturo Carpi [5].

Concerning Markoff numbers, it is worthwhile to mention the *Markoff numbers uniqueness conjecture*, or *Frobenius conjecture*, open since 1913. Among several formulations of this conjecture (see the book of Martin Aigner [1]), one is as follows: the map $w \mapsto 2 + |Pal \circ Antipal(w)|$, which is surjective from the set $a\{a, b\}^*$ onto the set of Markoff numbers $\neq 1, 2$, is also injective.

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2. Preliminaries

2.1. Christoffel words

Let p and q be two relatively prime integers, and let A and $B = A + (q, p)$ be two points in the discrete plane $\mathbb{Z} \times \mathbb{Z}$. The *Christoffel path* of slope p/q is the path, formed by elementary steps $[(x, y), (x + 1, y)]$ and $[(x, y), (x, y + 1)]$, from A to B, lying under the segment AB, and such that the polygon delimited by this segment and the path does not contain any integral point, except those lying on the path (see Fig. 1). The *Christoffel word* of slope p/q is the word w in the free monoid $\{a, b\}^*$, which represents the Christoffel path of slope p/q, where a represents a step $[(x, y), (x + 1, y)]$ and b represents a step $[(x, y), (x, y + 1)]$. For example, the Christoffel word of slope 4/7 is the word *aabaabaabab* (see Fig. 1). The previous definition is that of *lower* Christoffel words, but we call them simply Christoffel words in this article. In this section, we give several properties of these words (see [14] for more on this subject).

The number of occurrences of the letter a (resp. b) of the Christoffel word of slope p/q is q (resp. p). In particular, its length is $p + q$. A Christoffel word is *proper* if its length is > 1.

Denote by $\widetilde{w}$ the *reversal* of the word $w = a_1 \ldots a_n$, that is, $\widetilde{w} = a_n a_{n-1} \ldots a_1$. Recall that w is a *palindrome* if $\widetilde{w} = w$.

A proper Christoffel word has the form amb with m palindrome; the word m is called a *central word*. Central words are particular palindromes, with special properties.

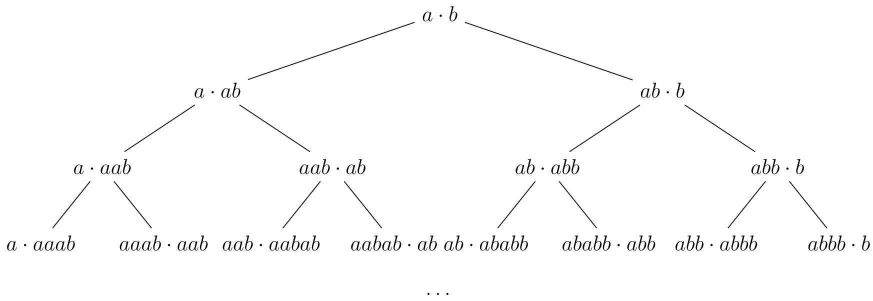


Fig. 2. Tree of Christoffel pairs.

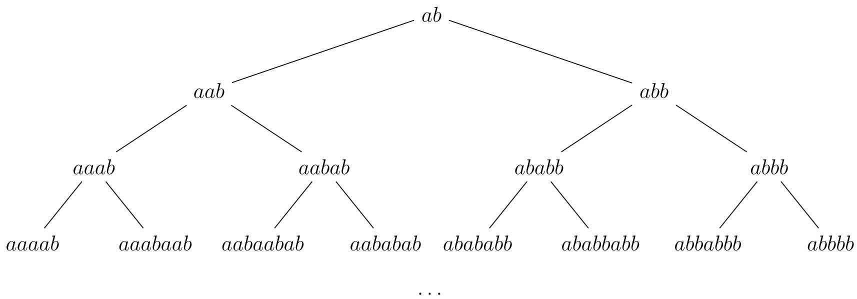


Fig. 3. Tree of Christoffel words.

The *standard factorization* of a proper Christoffel word w is a factorization $w = w_1 \cdot w_2$ such that the words w_1 and w_2 are Christoffel words. Jean-Pierre Borel and François Laubie [4] proved existence and uniqueness of this factorization. For example, the standard factorization of the Christoffel word *aabaabaabab* is *aab* · *aabaabab*. The triple (w_1, w, w_2) is called a *Christoffel triple*. Christoffel words have another remarkable factorization, called *palindromic factorization*: each proper Christoffel word is uniquely a product of two palindromes, as proved by Wai-Fong Chuan [6].

The standard factorization allows to construct the Christoffel words with the help of an infinite binary complete planar tree. The *tree of Christoffel pairs* is the tree with root labeled $a \cdot b$; any node labeled $w_1 \cdot w_2$ has as left and right children the nodes labeled $w_1 \cdot w_1 w_2$ and $w_1 w_2 \cdot w_2$. For example, the children of the node labeled *aab* · *ab* are labeled *aab* · *aabab* and *aabab* · *ab*. Fig. 2 represents the tree of Christoffel pairs.

Each Christoffel word, represented by its standard factorization, appears exactly once on the tree of Christoffel pairs.

A variant of the tree of Christoffel pairs is the *tree of Christoffel words*, obtained by replacing each label $w_1 \cdot w_2$ by $w = w_1 w_2$. The two tree are equivalents, since one recovers the label $w_1 \cdot w_2$ through the standard factorization of w.

It is also possible to construct directly the tree of Christoffel words as follows. The root is labeled *ab*. Let w be the label of some node, distinct from the root. Consider the path from this node towards the root.

(i) If w is not on one of the two extreme branches, then this path has at least one north-west step and one north-east step. Let w_1 be the label of the node at the end of the first north-west step, and w_2 be the label of the node at the end of the first north-east step.
(ii) If w is on the left (resp. right) extreme branch, then there is no north-west (resp. north-east) step on the path. Take then $w_1 = a$ and w_2 as in (*i*) (resp. w_1 as in (*i*) and $w_2 = b$).

Then $w = w_1 w_2$.

For example, in Fig. 3, consider $w = aabaabab$, then $w_1 = aab$ and $w_2 = aabab$.

2.2. Palindromization

We define now the palindromic closure and the iterated palindromization introduced by Aldo de Luca [8]. The *(right) palindromic closure* of a word w, denoted $w^{(+)}$, is the word $ps\widetilde{p}$ where s is the longest suffix of w which is a palindrome. For example, the palindromic closure of *abbab* is $(abbab)^{(+)} = abbabba$, with $p = ab$ and $s = bab$. The palindromic closure of a word is the shortest palindrome of which w is a prefix (Aldo de Luca Lemma 5 in [8]).

Define a function Pal from the free monoid into itself as follows: $Pal(1) = 1$ and for any word v and any letter x, $Pal(vx) = (Pal(v)x)^{(+)}$. The function Pal is called *iterated palindromization*. As an example, one has $Pal(abb) = ababa$. The function Pal is injective.

Justin [11] has characterized *Pal* by using morphisms, as follows. For any letter a in A, the endomorphism L_a of the free monoid A^* is defined by $L_a(a) = a$ and $L_a(x) = ax$ for any letter $x \neq a$. For any word $w = aw' \in A^*$ with $a \in A$, one has then $Pal(w) = Pal(aw') = L_a(Pal(w))a$.

The function *Pal* is a bijection from the free monoid $\{a,b\}^*$ onto the set of central words (see [8]). Thus, the function γ from $\{a,b\}^*$ into the set of Christoffel, defined by $v \mapsto aPal(v)b$, is a bijection. The word v is called the *directive word* of the Christoffel word $aPal(v)b$. The directive word v encodes the path, in the tree of Christoffel words, from the root towards the node x labeled $aPal(v)b$, with the rule that a represents a south-west step and b a south-east step. See Chapter 12 in [14] for these results and references.

We shall use the next result (see Corollary 3.2 of [2]). Let ν the homomorphism from $\{a,b\}^*$ into the multiplicative monoid $SL_2(\mathbb{N})$, defined by

$$\nu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \nu(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proposition 1. *Let $w = aPal(v)b$ be a Christoffel word and $w = w_1 w_2$ be its standard factorization. Then* $\nu(v) = \begin{pmatrix} |w_1|_a & |w_2|_a \\ |w_1|_b & |w_2|_b \end{pmatrix}$.

Iterated palindromization and Justin's formula are defined on any alphabet, but we shall use it only on the alphabet $\{a,b\}$.

2.3. Antipalindromization

The endomorphism E of the free monoid $\{a,b\}^*$ exchanges the letters, that is $E(a) = b$ and $E(b) = a$. Denote by $\widehat{w}$ the antiautomorphism $\widehat{w} = E(\widetilde{w}) = \widetilde{E(w)}$. An *antipalindrome* is a word such that $w = \widehat{w}$.

Using a similar approach to iterated palindromization, Aldo de Luca and Alessandro De Luca [9] have introduced *antipalindromization*: for any word w in $\{a,b\}^*$, there exists a shortest antipalindrome having w as prefix. The *(right) antipalindromic closure* of a word w, denoted by $w^{\oplus}$, is the word $ps\widehat{p}$ where s is the longest antipalindromic suffix of w. For example, one has $abbaba^{\oplus} = abbabaab$.

Define the endofunction *Antipal* of the free monoid $\{a,b\}^*$ as follows: $Antipal(vx) = (Antipal(v)x)^{\oplus}$. The function *Antipal* is called *iterated antipalindromization*. For example, $Antipal(abb) = abbaabbaab$.

There is a link between iterated palindromization and antipalindromization. Let θ be the Thue-Morse morphism, that is $\theta(a) = ab$ and $\theta(b) = ba$. Aldo de Luca and Alessandro De Luca [9] have shown that $Antipal = \theta \circ Pal$.

2.4. Markoff numbers

A *Markoff triple* is a multiset $\{x,y,z\}$ of positive integers satisfying the *Markoff equation*

$$x^2 + y^2 + z^2 = 3xyz.$$

For example $\{1,1,1\}, \{1,1,2\}$ and $\{1,2,5\}$ are Markoff triples. A Markoff triple is called *proper* if the three elements are distinct. A *Markoff number* is an element of a Markoff triple.

Let μ be the homomorphism from the free monoid $\{a,b\}^*$ into $SL_2(\mathbb{N})$ defined by

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let M be the function from $\{a,b\}^*$ into $\mathbb{N}$ defined by $M(w) = (\mu(w))_{12}$, that is, $M(w)$ is the 12-entry of the matrix $\mu(w)$. For example $M(aabab) = 194$. By results of Harvey Cohn [7], Enrico Bombieri [3] and the third author [12] (see also [14] page 21), the function which maps each Christoffel word w, with standard factorization $w_1 \cdot w_2$, onto the multiset $\{M(w_1), M(w_2), M(w)\}$ is a bijection from the set of proper Christoffel words onto the set of proper Markoff triples.

Replace, in the tree of Christoffel word, each Christoffel word by the Markoff triple, associated to it as above. The new tree is called the *tree of Markoff triples* (see Fig. 4). Since each Christoffel word appears exactly once on the tree of Christoffel words, each Markoff triple appears exactly once, as some equivalent (ordered) 3-tuple.

One may also construct the tree of Markoff triples directly from the Markoff equation. The root is $(1,5,2)$ and the recursive construction is shown in Fig. 5. See [1] and [14] for details.

3. Double palindromization and Markoff triples

Denote by $X = A^* \cup \{a^{-1}, b^{-1}\}$ the set obtained by adding to the free monoid the two symbols a^{-1} and b^{-1}. We define a mapping *Trio* from the free monoid into X^3, as follows ($k \geq 0$):

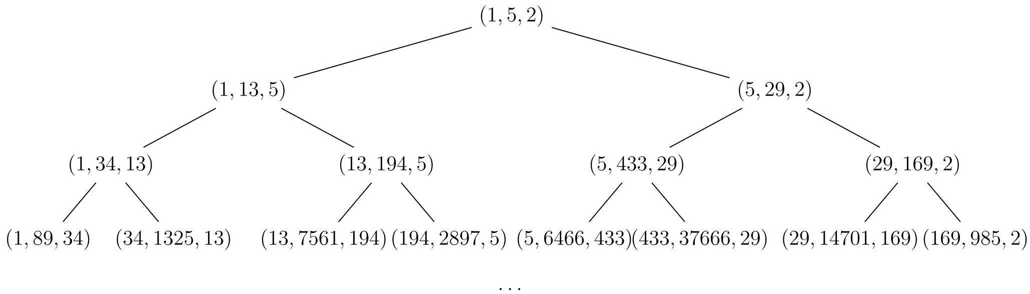


Fig. 4. Tree of Markoff triples.

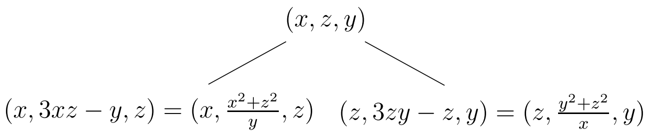


Fig. 5. Recursive construction of Markoff triples.

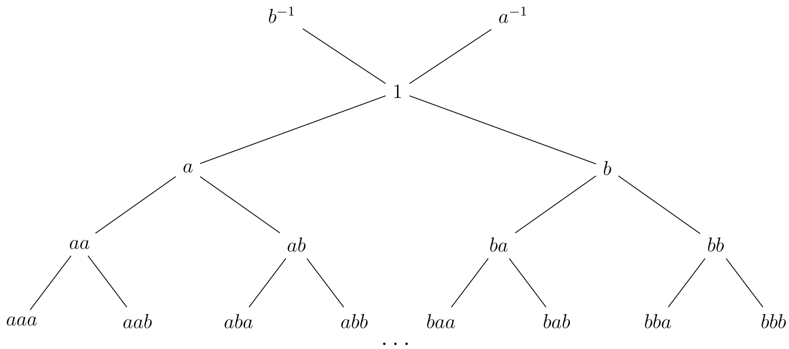


Fig. 6. Tree of $\{a, b\}^*$ with b^{-1} and a^{-1} as parents of the origin.

$$
Trio(w) = \begin{cases} (b^{-1}, a^k, a^{k-1}) & \text{if } w = a^k, \\ (b^{k-1}, b^k, a^{-1}) & \text{if } w = b^k, \\ (x, w, xba^k) & \text{if } w = xba^{k+1}, \\ (xab^k, w, x) & \text{if } w = xab^{k+1}. \end{cases}
$$

We may visualize this mapping by using the usual tree of the free monoid A^*, where we have added two parents b^{-1} and a^{-1} of the root, respectively left and right.

(i) Suppose first that w is not on the extreme branches. The path from this node towards the root has at least one north-west step and one north-east step. Let w_1 be the label of the node at the end of the first north-west path, and w_2 be the label of the node at the end of the first north-east step. We then have $Trio(w) = (w_1, w, w_2)$.

(ii) If w is on the extreme left (resp. right) branch, there is no north-west (resp. north-east) step. Then one takes $w_1 = b^{-1}$ and w_2 as in (i) (resp. $w_2 = a^{-1}$ and w_1 as in (i)). Then $Trio(w) = (w_1, w, w_2)$.

We illustrate this by two examples. If $w = abb$, see Fig. 6, the node after the first north-west step is ab and after the first north-east step, it is 1. Hence $Trio(w) = (ab, abb, 1)$. Now consider $w = bbb$. Its first north-west step is bb, and there is no north-east step. We then take a^{-1} as the third component, and thus $Trio(w) = (bb, bbb, a^{-1})$.

Note that $Trio$ is injective because w is the central component of $Trio(w)$. Note also that the tree-construction of $Trio$ is similar to the direct construction of the tree of Christoffel words, see Fig. 2. We use this in the proof below.

Recall that $\gamma(u) = aPal(u)b$. Extend it to X by $\gamma(a^{-1}) = b$ and $\gamma(b^{-1}) = a$. And now we extend it to X^3 naturally: $\gamma(w_1, w, w_2) = (\gamma(w_1), \gamma(w), \gamma(w_2))$. We then have the following result.

Lemma 1. *The function $\gamma \circ Trio$ is a bijection from the free monoid A^* onto the set of Christoffel triples.*

Proof. We show that $\forall v \in A^*$, $\gamma \circ Trio(v) = (w_1, w, w_2)$ with $w = \gamma(v) = w_1 w_2$ (standard factorization). This will imply the lemma, since $\gamma \mid A^*$ is a bijection from A^* onto the set of proper Christoffel words.

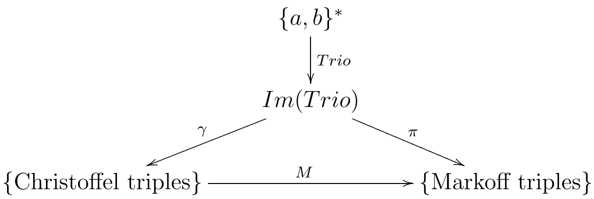


Fig. 7. The commutative diagram of Theorem 1.

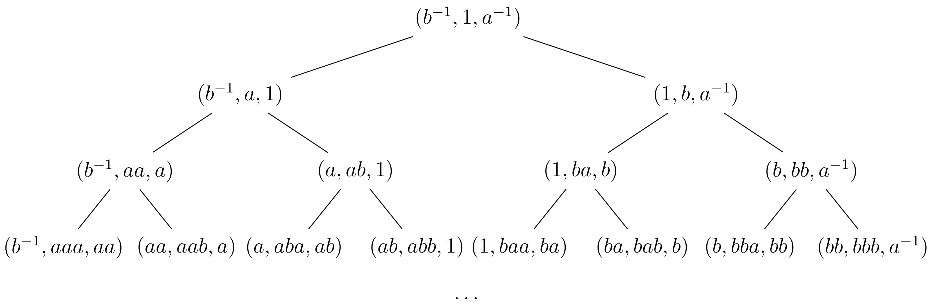


Fig. 8. Tree of Trio.

We know that γ is a bijection from X onto the set Christoffel words. When v is not on the extreme branches, by definition of $Trio$, the left and right components of $Trio(v)$ are respectively the words after the first north-west and first north-east step of the path from v towards the root in the tree of the free monoid. Thus, if $Trio(v) = (v_1, v, v_2)$, by the construction of the tree of Christoffel words in Section 2, we have $\gamma(v) = \gamma(v_1)\gamma(v_2)$.

Suppose now that v is on the left extreme branch, that is, $v = a^k$ for some $k \geq 0$. Then $Trio(v) = (b^{-1}, a^k, a^{k-1})$ and thus $\gamma \circ Trio(v) = (a, aa^k b, aa^{k-1}b)$ is as desired since $\gamma(a^k) = aa^k b$. Similarly for the right extreme branch.

This concludes the proof. □

We extend M to triples of words by the formula

$$M(w_1, w, w_2) = (M(w_1), M(w), M(w_2)).$$

Lemma 2. *The function $M \circ \gamma \circ Trio$ is a bijection from the free monoid onto Markoff triples.*

Proof. By Section 2, M is a bijection from the set of Christoffel triples onto the set of Markoff triples. By the previous result, $\gamma \circ Trio$ is a bijection from the free monoid onto the former set. □

Following [15], define now the function $\pi(v) = |Pal \circ \theta \circ Pal(av)| + 2$ from the free monoid into $\mathbb{N}$. The next result may be found in [15] (proof of Theorem 1) and in [13].

Lemma 3. *Let v be any word in $\{a, b\}^*$. Then $M \circ \gamma(v) = \pi(v)$.*

We extend π to X by $\pi(b^{-1}) = 1$ and $\pi(a^{-1}) = 2$, then to X^3 naturally: $\pi(w_1, w, w_2) = (\pi(w_1), \pi(w), \pi(w_2))$.

We can now state and prove the main result of this section.

Theorem 1. *The function $\pi \circ Trio$ is a bijection from the free monoid $\{a, b\}^*$ to Markoff triples.*

Proof. By Lemma 2, the function $M \circ \gamma \circ Trio$ is a bijection from the free monoid onto Markoff triples. Thanks to Lemma 3, we have equality between $M \circ \gamma$ and π. This implies the theorem. □

We may visualize the proof theorem through the commutative diagram in Fig. 7, where each arrow is a bijection.

The bijections may be seen as bijections between the trees in Figs. 4, 6 and 8. The function $Trio$ sends the tree of the free monoid onto the tree of Trio and the function π sends the latter onto the tree of Markoff triples.

A corollary of this theorem is the result obtained by Laurent Vuillon and the third author (see Theorem 1 in [15]).

Corollary 1. *For each word $v \in \{a, b\}^*$, the number $\pi(v)$ is a Markoff number $\neq$ 1,2. The mapping so defined is surjective.*

Note that the Frobenius conjecture is equivalent to the injectivity of π.

4. Study of Christoffel words of the form $a(Pal \circ Antipal(av))b$

One has $\pi(v) = 2 + |Pal \circ Antipal(av)|$. In other words, $\pi(v)$ is the length of the Christoffel word $a(Pal \circ Antipal(av))b$. In this section, we study these special Christoffel words.

4.1. Standard and palindromic factorizations

Christoffel words of the form $a(Pal \circ Antipal(av))b$ have a special standard factorization.

Theorem 2. *Let $v \in A^*$. The standard factorization of the Christoffel word $w = a(Pal \circ Antipal(av))b$ is*

$$aPal(ua)b \cdot aPal(u)b$$

with $u = (\theta \circ Pal(av))(ab)^{-1}$.

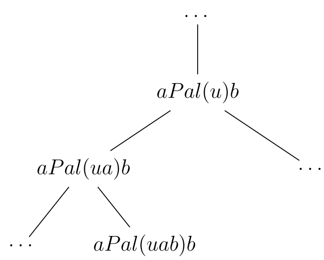
We shall see in the proof that $\theta \circ Pal(av)$ ends with ab.

As an example, take $v = b$. Then

$$a(Pal \circ Antipal(av))b = aababaababaababababaababaababab$$

and $u = (\theta \circ Pal(ab))(ab)^{-1} = abba$. Then the standard factorization of $a(Pal \circ Antipal(av))b$ is $aababaababaabababab \cdot aababaababab = aPal(ua)b \cdot aPal(u)b$ as predicted.

Proof. The word $\theta \circ Pal(av)$ ends by ab, since $Pal(av)$ is a palindrome beginning by a, hence ending by a and since $\theta(a) = ab$. Since $Antipal = \theta \circ Pal$, we have $w = a(Pal \circ \theta \circ Pal(av))b = aPal(uab)b$. Using the construction of the tree of Christoffel words, where u represents the path from the root towards $aPal(u)b$, one observes that the words $aPal(u)b$, $aPal(ua)b$, $aPal(uab)b$ are consecutive nodes in the tree and are placed as follows:



Hence the word uab is not on an extreme branch of the tree, since it contains both letters a and b. By construction of the tree of Christoffel words (see Section 2), the path represented by uab has both north-west and north-east steps. Let w_1 the node after the first north-west step, that is, $aPal(ua)b$ and w_2 the node after the first north-east step, that is $aPal(u)b$; then the standard factorization of w is $w_1 w_2 = aPal(ua)b \cdot aPal(u)b$. □

One obtains also the palindromic factorization, by applying Theorem 12.1.8 of [14].

Corollary 2. *The palindromic factorization of $w = a(Pal \circ Antipal(av))b$ is*

$$aPal(u)a \cdot bPal(ua)b.$$

Since we can describe precisely the standard factorizations of the words of the form $Pal \circ Antipal(av)$, we may improve the result of [13] on the Frobenius congruences.

Corollary 3. *Let $w = w_1 w_2 = aPal(v)b$ a proper Christoffel word with its standard factorization, and $(m_1, m, m_2) = (M(w_1), M(w), M(w_2))$ the corresponding Markoff triple. Then the unique solution $x \in \{0, 1, \ldots, m-1\}$ of the congruence $m_1 x \equiv m_2 \mod m$ (resp. $m_2 x \equiv m_1 \mod m$) is $x = |aPal(u)b|$ (resp. $x = |aPal(ua)b)|$) with $u = (\theta \circ Pal(av))(ab)^{-1}$.*

This follows from Corollary 9.2 in [13], using the explicit standard factorization given in Theorem 2.

4.2. Duality

We introduce a new notion of duality, distinct from the one in [2]. We are interested in this variant, since we shall see that the words of the form $a(Pal \circ Antipal(av))b$ are self-dual in this context. Thus there is a link between these words and the *harmonic* words of Arturo Carpi and Aldo de Luca [5].

Let w be a proper Christoffel word of slope $\frac{p}{q}$ and length $n = p + q$. The *dual Christoffel word* $w^\dagger$ is the Christoffel word of slope $\frac{p^\dagger}{q^\dagger}$, where $p^\dagger$ (resp. $q^\dagger$) is the unique integer in $\{0, 1, \dots, n-1\}$ satisfying the equation $pp^\dagger \equiv -1 \bmod n$ (resp. $qq^\dagger \equiv -1 \bmod n$).[1] For example, the dual of the Christoffel word $w = aabaabaabab$ of slope $\frac{4}{7}$ is the Christoffel word $w^\dagger = abbabbbabbb$ of slope $\frac{8}{3}$, because $4 \cdot 8 \equiv -1 \bmod 11$ and $7 \cdot 3 \equiv -1 \bmod 11$.

The dual of a Christoffel word always exists, since the integers p and q are relatively prime, hence also n, p and n, q. Thus the integers $p^\dagger$ and $q^\dagger$ exist and are relatively prime, hence their ratio is the slope of some Christoffel word. Moreover a Christoffel word and its dual have the same length. Indeed let $p^\dagger$ the integer n such that $pp^\dagger \equiv -1 \bmod n$ and $p^\dagger \in \{0, \dots, n-1\}$. Since $n = p + q$, we have $q(n - p^\dagger) \equiv -1 \bmod n$. But $n - p^\dagger \in \{0, \dots, n-1\}$. Hence $q^\dagger = n - p^\dagger$ and the Christoffel word and its dual have the same length.

Theorem 3. *Let $w = aPal(u)b$ be a proper Christoffel word. The dual Christoffel word of w is $w^\dagger = aPal(\widehat{u})b$.*

There exists a similar link between w and its dual w^* in the sense of the article [2]: for $w = aPal(u)b$ one has $w^* = aPal(\widetilde{u})b$, see [2] Proposition 3.1.

Proof. We could prove this result using continued fractions, following Arturo Carpi and Aldo de Luca [5]. However we use another approach, inspired from [2].

(1) Let $w = aPal(v)b$, $w' = aPal(\widehat{v})b$, $\nu(v) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $\nu(\widehat{v}) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. By Proposition 1, we have $|w|_a = a + b$, $|w|_b = c + d$, $|w'|_a = a' + b'$ and $|w'|_b = c' + d'$ par (1). In order to prove that $w' = w^\dagger$, it is enough to show that $a + b + c + d = a' + b' + c' + d' = n$ and that $(c + d)(c' + d') \equiv -1 \bmod n$.

(2) We prove that $\nu(\widehat{v}) = \nu(v)^t$, where M^t is matrix transposition, by induction on the length of v. When v is of length 1, the identity is true. Let v be of length at least 2; there exists a letter $x \in \{a, b\}$ and a word $v' \in \{a, b\}^*$ such that $v = v'x$. One has $\nu(\widehat{v}) = \nu(\widehat{v'x}) = \nu(\widehat{x})\nu(\widehat{v'}) = \nu(x)^t\nu(v')^t = (\nu(v')\nu(x))^t = \nu(v)^t$ by the induction hypothesis.

(3) We may therefore express $\nu(\widehat{v})$ in function of $\nu(v)$:

$$\nu(\widehat{v}) = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Hence the slope of w' is $\frac{c'+d'}{a'+b'} = \frac{b+d}{a+c}$. Moreover, $a + b + c + d = a' + b' + c' + d' = n$.

(4) We prove that $(c+d)(c'+d') \equiv -1 \bmod n$. More generally, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{N})$, then $(b+d)(c+d) \equiv -1 \bmod (a+b+c+d)$. Indeed, we have $ad - bc = 1$, hence $(b+d)(c+d) = bc + bd + dc + d^2 = ad - 1 + bd + dc + d^2 = -1 + d(a+b+c+d) \equiv -1 \bmod (a+b+c+d)$.

Hence the dual of $w = aPal(v)b$ is $w' = aPal(\widehat{v})b$, as desired. □

Corollary 4. *Let $w = aPal(v)b$ be a Christoffel word of slope $\frac{p}{q}$. The following statements are equivalent:*

1. *w is self-dual, that is, fixed by the involution $w \mapsto w^\dagger$;*
2. *v is an antipalindrome;*
3. *the slope of w is equal to the slope of $w^\dagger$;*
4. *$p^2 \equiv -1 \bmod n$;*
5. *$q^2 \equiv -1 \bmod n$.*

Corollary 4 means that self-dual words, in particular the words $Pal \circ Antipal(av)$, are harmonic in the sense of [5].

Corollary 5. *Christoffel words of the form $a(Pal \circ Antipal(av))b$ are self-dual.*

Proof. The image of the mapping *Antipal* is contained in the set of antipalindromes. By Corollary 4, $a(Pal \circ Antipal(av))b$ is self-dual. □

[1] The dual w^* of w in the sense of [2] is defined as the Christoffel word of slope $\frac{p^*}{q^*}$ with $pp^* \equiv 1 \bmod n$ and $qq^* \equiv 1 \bmod n$.

Note that besides our duality and the one of [2], there exists a third involution on Christoffel words: the one that inverts the slope. These four involutions (including the identity) act on the set of Christoffel words as a Klein group.

Declaration of competing interest

There is no conflict of interest.

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