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The Reciprocal of $\sum_{n\geq 0} a^n b^n$ for non-commuting a and b, Catalan numbers and non-commutative quadratic equations

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Note. This article is accompanied by the Maple package `NCFPS` downloadable from
`http://www.math.rutgers.edu/~zeilberg/tokhniot/NCFPS`

The aim of this paper is to describe the inversion of the sum $\sum_{n\geq 0} a^n b^n$ where a and b are non-commuting variables as a formal series in a and b. We show that the inversion satisfies a non-commutative quadratic equation and that the number of certain monomials in its homogeneous components equals a Catalan number. We also study general solutions of similar quadratic equations.

1. Inverting $\sum_{n\geq 0} a^n b^n$.

Our goal is to find an inverse of the series $\sum_{n\geq 0} a^n b^n$ where a and b are non-commuting variables. The answer to this question is given by the following theorem.

Let a, b, x be (completely!) **non-commuting** variables ("indeterminates"). Define a sequence of polynomials $d_n(a, b, x)$ ($n \geq 1$) recursively as follows:

$$d_1(a, b, x) = 1 \quad , \quad (1a)$$

$$d_n(a, b, x) = d_{n-1}(a, b, x)x + \sum_{k=2}^{n-1} d_{n-k}(a, b, x)\, a\, d_k(a, b, x)\, b \quad (n \geq 2) \quad .(1b)$$

Also define the sequence of polynomials $c_n(a, b, x)$ as follows:

$$c_n(a, b, x) = a\, d_n(a, b, x)\, b \quad (n \geq 1) \quad .$$

Theorem 1.

$$1 - \sum_{n=1}^{\infty} c_n(a, b, ab - ba) = \left(\sum_{n\geq 0} a^n b^n \right)^{-1} \quad .$$

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It follows immediately that the number of monomials in a, b *and* x *in the polynomial* $d_n(a,b,x)$ *is the* $(n-1)$*-th Catalan number. In particular,* $d_1 = 1$*,* $d_2 = x$*,* $d_3 = x^2 + axb$*,*

$$d_4 = x^3 + ax^2b + axbx + xaxb + a^2xb^2,$$

$$d_5 = x^4 + ax^2bx + axbx^2 + xaxbx + a^2xb^2x + x^2axb + axbaxb$$

$$+xax^2b + xaxb^2 + ax^3b + a^2x^2b^2 + a^2xbxb + axaxb^2 + a^3xb^3.$$

We will give an algebraic and a combinatorial proof of the theorem. A simple algebraic proof is based on two lemmas.

Lemma 2. *Let* S *be a formal series in* a *and* b *such that* $S = 1 + aSb$*. Observe that the inverse of* S *is of the form* $1 - C$ *where* $C = aDb$ *and the series* D *satisfies the equation*

$$D = 1 + D(x - ab) + DaDb \qquad (2)$$

and $x = ab - ba$*.*

Proof. We are looking for the inverse of S in the form $1 - C$ where $C = aDb$. We have

$$CS = (1 - S^{-1})S = S - 1 = aSb.$$

Hence

$$C(1 + aSb) = aSb,$$

$$C + CaSb = aSb,$$

$$aDb + aDbaSb = aSb.$$

So,

$$D + DbaS = S$$

and

$$D(1 + baS) = S$$

or

$$D(S^{-1} + ba) = 1.$$

It implies that

$$D(1 - C + ba) = 1$$

and

$$D = 1 + DaDb - Dba$$

which immediately implies equation (2). □

Lemma 3. *Let the degree of indeterminates* a *and* b *in equation (2) equal one and the degree of* x *equal two. Then the solution of equation (2) is given by formula*

$$D = \sum_{n \geq 1} d_n(a,b,x)$$

where polynomials $d_n(a,b,x)$ *satisfy equations (1).*

PROOF. Note that $D = \sum_{n=1}^{\infty} d_n$ where $d_n = d_n(a,b,x)$ are homogeneous polynomials in a and b of degree $2n-2$, $n = 1, 2, \ldots$.

The terms of degree 0 and 2 are: $d_1 = 1$ and $d_2 = x$.

Take the term of degree $2n-2$, $n \geq 3$:

$$d_n = d_{n-1}(x-ab)+\sum_{k=1}^{n-1} d_{n-k}ad_kb = d_{n-1}(x-ab)+d_{n-1}ab+d_1ad_{n-1}b+\sum_{k=2}^{n-2} d_{n-k}c_k =$$

$$= d_{n-1}x + ad_{n-1}b + \sum_{k=2}^{n-2} d_{n-k}c_k. \qquad \square$$

Let $S = \sum_{n\geq 0} a^n b^n$. Then S satisfies equation $S = 1 + aSb$ and Theorem 1 follows from Lemmas 2 and 3.

Combinatorial Proof: Consider the set of *lattice walks* in the 2D rectangular lattice, starting at the origin, $(0,0)$ and ending at $(n-1, n-1)$, where one can either make a *horizontal* step $(i,j) \to (i+1,j)$, (weight a), a *vertical* step $(i,j) \to (i,j+1)$, (weight b) or a diagonal step $(i,j) \to (i+1,j+1)$, (weight x), always staying in the region $i \geq j$, and where you can never have a horizontal step followed immediately by a vertical step. In other words, you may never venture to the region $i < j$, and you can never have the Hebrew letter Nun (alias the mirror-image of the Latin letter L) when you draw the path on the plane. The weight of a path is the product (in order!) of the weights of the individual steps.

For example, when $n = 2$ the only possible path is $(0,0) \to (1,1)$, whose weight is x.

When $n = 3$ we have two paths. The path $(0,0) \to (1,1) \to (2,2)$ whose weight is x^2 and the path $(0,0) \to (1,0) \to (2,1) \to (2,2)$ whose weight is axb.

When $n = 4$ we have five paths:

The path $(0,0) \to (1,1) \to (2,2) \to (3,3)$ whose weight is x^3,

the path $(0,0) \to (1,0) \to (2,1) \to (3,2) \to (3,3)$ whose weight is ax^2b,

the path $(0,0) \to (1,0) \to (2,1) \to (2,2) \to (3,3)$ whose weight is $axbx$,

the path $(0,0) \to (1,1) \to (2,1) \to (3,2) \to (3,3)$ whose weight is $xaxb$, and

the path $(0,0) \to (1,0) \to (2,0) \to (3,1) \to (3,2) \to (3,3)$ whose weight is a^2xb^2.

It is very well-known, and rather easy to see, that the number of such paths are given by the Catalan numbers $C(n-1)$, [**2**] `http://oeis.org/A000108` .

We claim that the *weight-enumerator* of the set of such walks equals $d_n(a,b,x)$. Indeed, since the walk ends on the diagonal, at the point $(n-1, n-1)$, the last step must be either a diagonal step

$$(n-2, n-2) \to (n-1, n-1) \quad ,$$

whose weight-enumerator, by the inductive hypothesis is $d_{n-1}(a,b,x)x$, or else let k be the smallest integer such that the walk passed through $(n-k-1, n-k-1)$ (i.e. the penultimate encounter with the diagonal). Note that k can be anything between 2 and $n-1$. The weight-enumerator of the set of paths from $(0,0)$ to $(n-k-1, n-k-1)$ is $d_{n-k}(a,b,x)$, and the weight-enumerator of the set of paths from $(n-k-1, n-k-1)$ to $(n-1, n-1)$ that never touch the diagonal, is $ad_k(a,b,x)b$. So the weight-enumerator is $d_{n-k}(a,b,x)\, a\, d_k(a,b,x)b$ giving the above recurrence for $d_n(a,b,x)$.

It follows that $c_n(a, b, x) = ad_n(a, b, x)b$ is the weight-enumerator of all paths from $(0, 0)$ to (n, n) as above with the additional property that except at the beginning $((0, 0))$ and the end $((n, n))$ they always stay **strictly** below the diagonal.

Now what does $c_n(a, b, ab - ba)$ weight-enumerate? Now there is a new rule in Manhattan, "no shortcuts", one may not walk diagonally. So every diagonal step $(i, j) \to (i+1, j+1)$ must decide whether to go first horizontally, and then vertically $(i, j) \to (i + 1, j) \to (i + 1, j + 1)$, replacing x by ab, or to go first vertically, and then horizontally $(i, j) \to (i, j + 1) \to (i + 1, j + 1)$, replacing x by $-ba$.

This has to be decided, independently for each of the diagonal steps that formerly had weight x. So a path with r diagonal steps gives rise to 2^r new paths with sign $(-1)^s$ where s is the number of places where it was decided to go through the second option.

So $c_n(a, b, ab - ba)$ is the weight-enumerators of pairs of paths $[P, K]$ where P is the original path featuring a certain number of diagonal steps r, and K is one of its 2^r "children", paths with only horizontal and vertical steps, and weight $\pm weight(P)$, where we have a plus-sign if an even number of the r diagonal steps became *vertical-then-horizontal* (i.e. ba) and a minus-sign otherwise.

As we look at the weights of the children K, sometimes we have the same path coming from different parents. Let's call a pair $[P, K]$ *bad* if the path P has a "ba" *strictly-under* the diagonal, i.e. a "vertical step followed by a horizontal step" that does not touch the diagonal. Write K as $K = w_1(ba)^s w_2$ where w_1 does not have any sub-diagonal ba's and s is as large as possible. Then the parent must be either of the form $P = W_1 x^s W_2$ where the x^s corresponds to the $(ba)^s$, or of the form $P' = W_1 b x^{s-1} a W_2$. In the former case attach $[W_1 x^s W_2, K]$ to $[W_1 b x^{s-1} a W_2, K]$ and in the latter case vice-versa. This is a weight-preserving and **sign-reversing** involution among the bad pairs, so they all kill each other.

It remains to weight-enumerate the *good pairs*. It is easy to see that the good pairs are pairs $[P, K]$ where K has the form $K = a^{i_1} b^{i_1} a^{i_2} b^{i_2} \dots a^{i_s} b^{i_s}$ for some $s \geq 1$ and integers $i_1, \dots, i_s \geq 1$ summing up to n (this is called a *composition* of n). It is easy to see that for each such K, (coming from a good pair $[P, K]$)there can only be **one** possible *parent* P. The sign of a good pair

$$[P, a^{i_1} b^{i_1} a^{i_2} b^{i_2} \dots a^{i_s} b^{i_s}] \quad ,$$

is $(-1)^{s-1}$, since it touches the diagonal $s - 1$ times, and each of these touching points came from an x that was turned into $-ba$.

So $1-\sum_{n=1}^{\infty} c_n(a, b, ab-ba)$ turned out to be the sum of all the weights of compositions (vectors of positive integers) $(i_1, \dots, i_s)$ with the weight $(-1)^s a^{i_1} b^{i_1} \cdots a^{i_s} b^{i_s}$ over *all* compositions, but the same is true of

$$\left(\sum_{n \geq 0} a^n b^n \right)^{-1} = \left(1 + \sum_{n \geq 1} a^n b^n \right)^{-1} = 1 + \sum_{s=1}^{\infty} (-1)^s \left(\sum_{n \geq 1} a^n b^n \right)^s \quad .$$

QED!

2. Inversion of $1 - aDb$ in the general case

The following is a variant of of path's model used in Section 1. Call *Dyck path* a path that starts at the origin, ends on the x-axis, that uses the steps $(1, 1)$ (denoted by a) and $(1, -1)$ (denoted by b), and that never goes below the x-axis. It is coded by a *Dyck word*, e.g. $aaababbabb$. Formally, a Dyck word has as many a's than b's, and each prefix of it has at least as many a's as b's.

If we replace, in each Dyck word, each occurrence of ab by a letter x, and sum all these words, then we obtain the series $D = \sum_{n\geq 1} d_n$ described in Section 1.

If we replace each ab by a letter x, except those at level 0, then we obtain the series

$$1 + aUb = 1 + \sum_{n\geq 1} au_n b.$$

For a series Z set $Z^* := (1 - Z)^{-1}$. Then

$$(aDb)^* = 1 + aUb.$$

THEOREM 4. *One has the equation*

$$U = (1 + aUb)(1 + (x - ab + ba)U)$$

that completely defines U.

PROOF. We have $(1 - aDb)^{-1} = 1 + aUb$, thus $1 - aDb = (-aUb)^*$.
The defining equation for D is

$$D = 1 + (x - ab + aDb)D \qquad (3)$$

which is a symmetric version of equation (2); it follows from the Dyck path model, by writing $D = 1 + (x + a(D - 1)b)D$. Let $D = 1 + d_1(a, b, x) + d_2(a, b, x) + \dots$ and polynomials $d_n(a, b, x)$ that satisfy equations (1) without any assumptions on x.

We have

$$1 - aDb = (-aUb)^* = 1 - aUb + (aUb)^2 - (aUb)^3 + \dots \quad .$$

Therefore,

$$\begin{aligned} aDb = aUb - (aUb)^2 + (aUb)^3 - \dots &= a(U - UbaU + UbaUbaU - \dots)b \\ &= aU(1 - baU + (baU)^2 - \dots)b \end{aligned}$$

and

$$D = U(-baU)^*.$$

Note that (3) implies

$$U(-baU)^* = 1 + (x - ab + aU(-baU)^*b)U(-baU)^*$$

therefore,

$$\begin{aligned} U &= 1 + baU + (x - ab)U + aU(-baU)^*bU \\ &= 1 + (x - ab + ba)U + aU(1 - baU + baUbaU - \dots)bU \\ &= 1 + (x - ab + ba)U + aUbU - aUbaUbU + aUbaUbaUbU - \dots \\ &= 1 + (x - ab + ba)U + (-aUb)^*aUbU. \end{aligned}$$

Hence

$$(1 + aUb)U = 1 + aUb + (1 + aUb)(x - ab + ba)U + aUbU$$

and

$$\begin{aligned} U &= 1 + aUb + (1 + aUb)(x - ab + ba)U \\ &= (1 + aUb)(1 + (x - ab + ba)U). \end{aligned}$$

□

REMARK 5. If we put $x = ab - ba$ in the last equation, then $U = 1 + aUb$ which implies $U = \sum_{n\geq 1} a^{n-1}b^{n-1}$ and $1 + aUb = \sum_{n\geq 0} a^n b^n$.

Note that Theorem 4 does not imply that all coefficients in U as series in a, b and x are positive. However, simple computations show that the inversion of the series $1 - aDb$ is written in the form

$$1 + au_1b + au_2b + ...$$

where the degree of u_n is $2n - 2$, $n \geq 1$ and

$$\begin{aligned} u_1 &= 1, \\ u_2 &= ba + x, \\ u_3 &= (ba)^2 + xba + bax + axb + x^2, \\ u_4 &= (ba)^3 + x(ba)^2 + baxba + (ba)^2x + a^2xb + axb^2a + ba^2xb \\ &\quad + x^2ba + xbax + bax^2 + ax^2b^2 + axbx + xaxb + x^3, \end{aligned}$$

and so on. The positivity follows from the path interpretation at the beginning of the section.

PROBLEM 6. How to write a recurrence relations on u_n similar to relations (1). It must imply that the number of terms for u_n is the n-th Catalan number. It also must show that if $x = ab - ba$ then $u_n = a^{n-1}b^{n-1}$.

We may set $x = 1$ and get

$$u_1 = 1, \quad u_2 = ba + 1, \quad u_3 = (ba)^2 + 2ba + ab + 1,$$
$$u_4 = (ba)^3 + 3(ba)^2 + ab^2 + ba^2b + a^2b^2 + 3ba + 3ab + 1.$$

PROBLEM 7. How to describe polynomials u_n for this and other specializations? Any relations with known polynomials?

3. The Quasideterminant of a Jacobi Matrix

In this section we discuss solutions of noncommutative quadratic equation (2) using quasideterminants. Recall [**1**] that quasideterminant $|A|_{pq}$ of the matrix $A = (a_{ij})$, $i, j = 1, 2, \ldots$ is defined as follows. Let A^{pq} be the submatrix of A obtained from A by removing its p-th row and q-th column. Denote by r_p and c_q be the p-th row and the q-th column of A with element a_{pq} removed. Assume that matrix A^{pq} is invertible. Then

$$|A|_{pq} := a_{pq} - r_p(A^{pq})^{-1}c_q.$$

Let now $A = (a_{ij})$, $i, j \geq 1$ be a Jacobi matrix, i.e. $a_{ij} = 0$ if $|i - j| > 1$. Set $T = I - A$, where I is the infinite identity matrix. Recall that

$$|T|_{11}^{-1} = 1 + \sum a_{1j_1}a_{j_1j_2}a_{j_2j_3}\ldots a_{j_k1}$$

where the sum is taken over all tuples $(j_1, j_2, \ldots, j_k)$, $j_1, j_2, \ldots, j_k \geq 1$, $k \geq 1$.

Also,

$$|T|_{11} = 1 - a_{11} - \sum a_{1j_1}a_{j_1j_2}a_{j_2j_3}\ldots a_{j_k1}$$

where the sum is taken over all tuples $(j_1, j_2, \ldots, j_k)$, $j_1, j_2, \ldots, j_k > 1$, $k \geq 1$.

Assume that the degree of all diagonal elements a_{ii} is two and the degree of all elements a_{ij} such that $i \neq j$ is one. Then

$$|T|_{11}^{-1} = 1 + \sum_{n\geq 1} t_n \qquad (3)$$

where t_n is homogeneous polynomial of degree $2n$ in variables a_{ij}.

In particular,

$$t_1 = a_{11} + a_{12}a_{21},$$

$$t_2 = a_{11}^2 + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11} + a_{12}a_{22}a_{21} + (a_{12}a_{21})^2 + a_{12}a_{23}a_{32}a_{21}.$$

Note that each monomial corresponds, in a one-to-one way, to a "Schröder walk"
[2] `http://oeis.org/A006318`, hence:

PROPOSITION 8. *The number of monomials of t_n is the n-th Large Schröder Number.*

If we set $a_{11} = 0$ we get walks obviously counted by the "little" Schröder numbers
[2] `http://oeis.org/A001003`, hence:

PROPOSITION 9. *Set $a_{11} = 0$. Then the number of monomials in each t_n is $A001003[n]$.*

Let now a, x, b be formal variables, the degree of a and b is one and the degree of x is two. Set $a_{ii} = x - ab$, $a_{i,i+1} = a, a_{i+1,i} = b$ for all i. By the definition of quasideterminants, we have

$$|T|_{11} = 1 - x + ab - a|T|_{11}^{-1}b.$$

Denote $|T|_{11}^{-1}$ by D. Then last equation can be written as

$$D^{-1} = 1 - x + ab - aDb$$

or

$$D = 1 + D(x - ab) + DaDb$$

which is exactly our equation (2).

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