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Return words of linear involutions and fundamental groups

VALÉRIE BERTHÉ†, VINCENT DELECROIX‡, FRANCESCO DOLCE§, DOMINIQUE PERRIN§, CHRISTOPHE REUTENAUER¶ and GIUSEPPINA RINDONE§

† *CNRS, LIAFA, Université Paris Diderot, Paris 7 – Case 7014, F-75205 Paris Cedex 13, France*
‡ *LaBRI, UMR 5800, Bâtiment A30, 351, cours de la Libération, 33405 Talence Cedex, France*
§ *Université Paris-Est Marne-la-Vallée, Laboratoire d'informatique Gaspard-Monge, UMR 8049 CNRS, 5 Bd Descartes, Champs-sur-Marne, F-77454 Marne-la-Vallée Cedex 2, France*
(e-mail: dominique.perrin@esiee.fr)
¶ *Département de mathématiques, Université du Québec à Montréal, C.P. 8888, Succursale Centre-Ville Montréal, Québec H3C 3P8, Canada*

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Abstract. We investigate the shifts associated with natural codings of linear involutions. We deduce, from the geometric representation of linear involutions as Poincaré maps of measured foliations, a suitable definition of return words which yields that the set of return words to a given word is a symmetric basis of the free group on the underlying alphabet, A. The set of return words with respect to a subgroup of finite index G of the free group on A is also proved to be a symmetric basis of G.

1. *Introduction*

A linear involution is an injective piecewise isometry defined on a pair of intervals. This generalization of the notion of interval exchange allows one to work with non-orientable foliations (see e.g. [**12**] and [**8**, Convention 2.2]). Linear involutions were introduced by Danthony and Nogueira in [**12**, **13**], generalizing interval exchanges with flip(s) [**24**, **25**] (these are interval exchange transformations which reverse orientation in at least one interval). They extended the notion of Rauzy induction (introduced in [**26**]) to these transformations. The study of linear involutions was later developed by Boissy and Lanneau in [**8**] in connection with the classification of connected components of exceptional strata of meromorphic quadratic differentials. Note that various generalizations of interval exchanges exist: for example, pseudogroups of isometries [**20**] and interval identification systems [**27**].

In the present paper, we study natural codings of linear involutions in the spirit of our previous papers on Sturmian sets [**1**] and their generalizations as tree sets [**3**–**6**]. A tree set is a factorial set of words that all satisfy a combinatorial condition expressed in terms of the possible extensions of these words within the tree set: the condition is that the extension graph of each word is a tree, with this graph describing the possible extensions of a word in the language on the left and on the right. Tree sets encompass the languages of classical shifts of zero entropy like the ones generated by Sturmian words, Arnoux–Rauzy words or natural codings of interval exchanges. Note, however, that all these shifts display various behaviors in terms of spectral properties (they can be weakly mixing, or they can have pure discrete spectrum).

Tree sets have interesting properties relating to free groups and symbolic dynamics. In particular, tree sets allow one to exhibit bases of the free group, or of subgroups of the free group. Indeed, in a uniformly recurrent tree set, the sets of return words to a given word are bases of the free group on the alphabet [**6**]. Moreover, maximal bifix codes that are included in uniformly recurrent tree sets provide bases of subgroups of finite index of the free group [**4**].

These properties thus hold for tree sets associated with regular interval exchange sets. Observe that the fact that return words are bases of the free group can either be deduced combinatorially from the property that interval exchanges yield tree sets [**3**] or, as we will show here, from the geometric interpretation of interval exchanges as Poincaré sections of linear flows on translation surfaces: return words provide bases of the fundamental group of the associated surface.

The natural coding of a linear involution is the set of factors of the infinite words that encode the sequences of subintervals met by the orbits of the transformation. They are defined on an alphabet A whose letters and their inverses index the intervals exchanged by the involution. A natural coding is thus a subset of the free group F_A on the alphabet A. One of our motivations for working with natural involutions is that they provide a first step for extending the combinatorial study of tree sets, done in the framework of monoids, to free groups. A further motivation is to provide a classification of free group automorphisms associated with tree sets according to the botany developed in [**10**] for fully irreducible outer automorphisms of the free group.

We extend most of the properties proved for uniformly recurrent tree sets to natural codings of linear involutions, and thus to natural codings of interval exchanges. The extension is not immediate. If linear involutions have a geometric interpretation as Poincaré maps of measured foliations, one has to modify the definition of return words in order to make it consistent with the notion of a Poincaré map of a foliation. We thus consider return words to the set $\{w, w^{-1}\}$ and we consider a truncated version of them, that we call *mixed return words*. We also have to replace the basis of a subgroup by its symmetric version containing the inverses of its elements, called a symmetric basis. The free group is then obtained as the fundamental group of a compact surface from which a finite number of points are removed, and linear involutions are seen as Poincaré sections of measured foliations of the surface. The return words to a given word can be seen as different ways of choosing a section.

We prove that if a language S is the natural coding of a linear involution T without connections on the alphabet A, the following holds.

- The set of mixed return words to a given word w (recall that they are defined with respect to the set $\{w, w^{-1}\}$) in S is a symmetric basis of the free group on A (Theorem 6.4).
- Let G be a subgroup of finite index of the free group F_A. The set of prime words in S with respect to G is a symmetric basis of G (Theorem 6.9). By prime words in S with respect to G, we mean the non-empty words in $G \cap S$ without a proper non-empty prefix in $G \cap S$.

Even if the proofs provided here concerning the algebraic properties of return words are of a topological and geometric flavor, these properties hold in a wider combinatorial context through the notion of specular sets and specular groups, where the present geometric background does *a priori* not exist. Extensions of Theorems 6.4 and 6.9 are proved to hold in this context in [**2**]. We also emphasize, as part of our motivation, the great importance of return words as a tool for symbolic dynamics, and its relationship with other structures. For instance, they are closely related to induction and renormalization, they allow the characterization of substitutive words [**15**], they provide spectral information through eigenvalues (see, e.g., [**9**]), and they yield so-called S-adic representations [**16**, **17**].

This paper is organized as follows. In §2, we recall the concepts of words, free groups and graphs. Linear involutions are defined in §3. We also recall that, by a result of [**8**], a non-orientable linear involution without connections is minimal. In §4, we provide the necessary geometric background on natural involutions, while in §5, we focus on the symbolic properties of their natural codings and we introduce the definition of return words, even letters and the even group. The geometric and topological proofs of the main results on return words for natural codings of linear involutions are given in §6.

2. *Words, free groups and laminary sets*

In this section, we introduce the notion of sets of words and free groups.

Let A be a finite non-empty alphabet and let A^* be the set of all words on A. We let 1 or ε denote the empty word. A set of words is said to be *factorial* if it contains the factors of its elements.

The notation a^{-1} will be interpreted as an inverse in the free group F_A on A. We also use the notation $\bar{a}$ instead of a^{-1}.

A set of reduced words on the alphabet $A \cup A^{-1}$ is said to be *symmetric* if it contains the inverses of its elements. Let X^* be the submonoid of $(A \cup A^{-1})^*$ generated by X without reducing the products. If X is symmetric, the subgroup of F_A generated by X is the set obtained by reducing the words of X^*.

Definition 2.1. (Symmetric basis) If X is a basis of a subgroup H of F_A, the set $X \cup X^{-1}$ is called a *symmetric basis* of H.

In particular, $A \cup A^{-1}$ is a symmetric basis of F_A. Note that a symmetric basis $X \cup X^{-1}$ is not a basis of H but that any $w \in H$ can be written uniquely $w = x_1 x_2 \cdots x_n$ with $x_i \in X \cup X^{-1}$ and $x_i x_{i+1}$ is not equivalent to 1 for $1 \leq i \leq n-1$. We recall that, by *Scheier's*

Formula, any basis of a subgroup of index d of a free group on k symbols has $d(k-1)+1$ elements. Hence, if Y is a symmetric basis of a subgroup of index d in a free group on k symbols, then $\mathrm{Card}(Y) = 2d(k-1)+2$.

The next definition follows [**11**, **23**].

Definition 2.2. (Laminary set) A symmetric factorial set of reduced words on the alphabet $A \cup A^{-1}$ is called a *laminary set* on A.

A laminary set S is called *semi-recurrent* if, for any $u, w \in S$, there is $v \in S$ such that $uvw \in S$ or $uvw^{-1} \in S$. Likewise, it is said to be *uniformly semi-recurrent* if it is right extendable and if, for any word $u \in S$, there is an integer $n \geq 1$ such that for any word w of length n in S, u or u^{-1} is a factor of w. A uniformly semi-recurrent set is semi-recurrent.

Following again the terminology of [**11**], we say that a laminary set S is *orientable* if there exist two factorial sets S_+, S_- such that $S = S_+ \cup S_-$ with $S_+ \cap S_- = \{\varepsilon\}$ and for any $x \in S$, one has $x \in S_-$ if and only if $x^{-1} \in S_+$. Note that if S is a semi-recurrent orientable laminary set, then the sets S_+, S_-, as above, are unique (up to their interchange). The sets S_+, S_- are called the *components* of S. Moreover a uniformly recurrent and orientable laminary set is a union of two uniformly recurrent sets. Indeed, S_+ and S_- are uniformly recurrent.

3. *Linear involutions*

In this section, we define linear involutions, which are a generalization of interval exchange transformations. First, we give the basic definitions including generalized permutation and length data, and then discuss minimality for involutions in relation to the notion of connection.

3.1. *Definition.*

Let A be an alphabet with k elements.

We consider two copies $I \times \{0\}$ and $I \times \{1\}$ of an open interval I of the real line and we define $\hat{I} = I \times \{0, 1\}$. We call the sets $I \times \{0\}$ and $I \times \{1\}$ the two *components* of $\hat{I}$. We consider each component as an open interval.

A *generalized permutation* on A of type (ℓ, m), with $\ell + m = 2k$, is a bijection $\pi : \{1, 2, \ldots, 2k\} \to A \cup A^{-1}$. We represent it by a two line array

$$\pi = \begin{pmatrix} \pi(1)\pi(2)\cdots\pi(\ell) \\ \pi(\ell+1)\cdots\pi(\ell+m) \end{pmatrix}.$$

A *length data* associated with (ℓ, m, π) is a non-negative vector $\lambda \in \mathbb{R}_+^{A \cup A^{-1}} = \mathbb{R}_+^{2k}$ such that

$$\lambda_{\pi(1)} + \cdots + \lambda_{\pi(\ell)} = \lambda_{\pi(\ell+1)} + \cdots + \lambda_{\pi(2k)} \quad \text{and} \quad \lambda_a = \lambda_{a^{-1}} \quad \text{for all } a \in A.$$

We consider a partition of $I \times \{0\}$ (minus $\ell - 1$ points) into ℓ open intervals $I_{\pi(1)}, \ldots, I_{\pi(\ell)}$ of lengths $\lambda_{\pi(1)}, \ldots, \lambda_{\pi(\ell)}$ and a partition of $I \times \{1\}$ (minus $m-1$ points) into m open intervals $I_{\pi(\ell+1)}, \ldots, I_{\pi(\ell+m)}$ of lengths $\lambda_{\pi(\ell+1)}, \ldots, \lambda_{\pi(\ell+m)}$. Recall that $2k = \ell + m$. Let Σ be the set of $2k-2$ *division points* separating the intervals I_a for $a \in A \cup A^{-1}$.

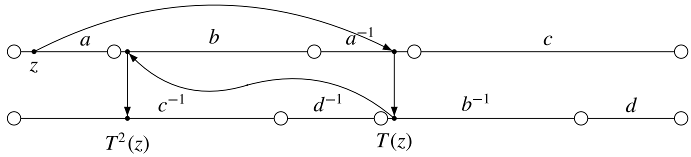

FIGURE 3.1. A linear involution.

The *linear involution* on I relative to these data is the map $T = \sigma_2 \circ \sigma_1$ defined on the set $\hat{I} \backslash \Sigma$, formed of $\hat{I}$ minus $2k - 2$ points, which is the composition of two involutions defined as follows.

(i) The first involution σ_1 is defined on $\hat{I} \backslash \Sigma$. It is such that for each $a \in A \cup A^{-1}$, its restriction to I_a is either a translation or a symmetry from I_a on to $I_{a^{-1}}$. Since σ_1 is an involution, its respective restrictions to I_a and $I_{a^{-1}}$ are of the same nature, being either a translation or a symmetry. Thus, there are real numbers α_a such that for any $(x, \delta) \in I_a$, one has $\sigma_1(x, \delta) = (x + \alpha_a, \gamma)$ in the first case, and $\sigma_1(x, \delta) = (-x + \alpha_a, \gamma)$ in the second case (with $\gamma \in \{0, 1\}$).

(ii) The second involution exchanges the two components of $\hat{I}$. It is defined for $(x, \delta) \in \hat{I}$ by $\sigma_2(x, \delta) = (x, 1 - \delta)$. The image of z by σ_2 is called the *mirror image* of z.

We also say that T is a *linear involution on* I *relative to the alphabet* A or that it is a k*-linear involution* to express the fact that the alphabet A has k elements.

Example 3.1. Let $A = \{a, b, c, d\}$ and

$$\pi = \begin{pmatrix} a & b & a^{-1} & c \\ c^{-1} & d^{-1} & b^{-1} & d \end{pmatrix}.$$

Let T be the 4-linear involution corresponding to the length data represented in Figure 3.1. We represent $I \times \{0\}$ above $I \times \{1\}$ with the assumption that the restriction of σ_1 to I_a and I_d is a symmetry while its restriction to I_b, I_c is a translation. We indicate on the figure the effect of the transformation T on a point z located in the left part of the interval I_a. The point $\sigma_1(z)$ is located in the right part of $I_{a^{-1}}$, and the point $T(z) = \sigma_2\sigma_1(z)$ is just below on the left of $I_{b^{-1}}$. Next, the point $\sigma_1 T(z)$ is located on the left part of I_b and the point $T^2(z)$ just below.

Thus, the concept of linear involution is an extension of the notion of interval exchange transformation in the following sense. Assume that $\ell = m = k$, that $A = \{\pi(1), \ldots, \pi(k)\}$, and that the restriction of σ_1 to each subinterval is a translation. Then, the restriction of T to $I \times \{0\}$ is an interval exchange (and so is its restriction to $I \times \{1\}$ which is the inverse of the first one). Thus, in this case, T is a pair of mutually inverse interval exchange transformations.

It is also an extension of the notion of interval exchange with flip(s) [**24**, **25**]. Assume again that $\ell = m = k$ and that $A = \{\pi(1), \ldots, \pi(k)\}$, but now that the restriction of σ_1 to at least one subinterval is a symmetry. Then the restriction of T to $I \times \{0\}$ is an interval exchange with flip(s).

Note that, in this paper, we consider interval exchange transformations defined by a partition of an open interval minus a finite number of points in a finite number of open

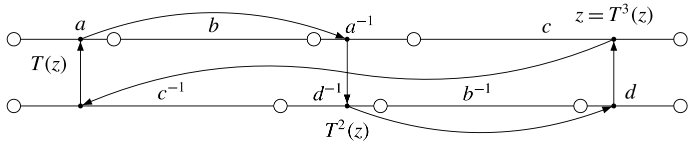

FIGURE 3.2. A non-coherent linear involution.

intervals. The usual concept of interval exchange transformation uses a partition of a semi-interval in a finite number of semi-intervals. One arrives at the usual idea of interval exchange transformation on a semi-interval by attaching to each open interval its left endpoint.

A linear involution T is a bijection from $\hat{I}\backslash\Sigma$ onto $\hat{I}\backslash\sigma_2(\Sigma)$. Since σ_1, σ_2 are involutions and $T = \sigma_2 \circ \sigma_1$, the inverse of T is $T^{-1} = \sigma_1 \circ \sigma_2$.

The set of *singular points* of T is defined as the set Σ of division points of T and their mirror images are the singular points of T^{-1}. Note that these singular points z may be 'false' singularities (that is, false discontinuities), in the sense that T can have a continuous extension to an open neighborhood of z.

Two particular cases of linear involutions deserve attention.

Definition 3.2. (Non-orientable linear involution) A linear involution T on the alphabet A relative to a generalized permutation π of type (ℓ, m) is said to be *non-orientable* if there are indices $i, j \le \ell$ such that $\pi(i) = \pi(j)^{-1}$ (and thus indices $i, j \ge \ell + 1$ such that $\pi(i) = \pi(j)^{-1}$). In other words, there is some $a \in A \cup A^{-1}$ for which I_a and $I_{a^{-1}}$ belong to the same component of $\hat{I}$. Otherwise, T is said to be *orientable*.

Definition 3.3. (Coherent linear involution) A linear involution $T = \sigma_2 \circ \sigma_1$ on I relative to the alphabet A is said to be *coherent* if, for each $a \in A \cup A^{-1}$, the restriction of σ_1 to I_a is a translation if and only if I_a and $I_{a^{-1}}$ belong to distinct components of $\hat{I}$.

Example 3.4. The linear involution of Example 3.1 is coherent. Let us now consider the linear involution T which is the same as in Example 3.1, but such that the restriction of σ_1 to I_c is a symmetry. Thus, T is not coherent. We assume that $I =]0, 1[$, that $\lambda_a = \lambda_d$, that $1/4 < \lambda_c < 1/2$ and that $\lambda_a + \lambda_b < 1/2$. Let $z = 1/2 + \lambda_c$ (see Figure 3.2). We then have $T^3(z) = z$, showing that T is not minimal. Indeed, since $z \in I_c$, we have $T(z) = 1 - z = 1/2 - \lambda_c$. Since $T(z) \in I_a$, we have $T^2(z) = (\lambda_a + \lambda_b) + (\lambda_a - 1 + z) = z - \lambda_c = 1/2$. Finally, since $T^2(z) \in I_{d^{-1}}$, we obtain $1 - T^3(z) = T^2(z) - \lambda_c = 1 - z$ and thus $T^3(z) = z$.

Linear involutions which are orientable and coherent correspond to interval exchange transformations, whereas orientable but non-coherent linear involutions are interval exchanges with flip(s).

Orientable linear involutions correspond to orientable laminations (see §4), whereas coherent linear involutions correspond to orientable surfaces. Thus, coherent non-orientable involutions correspond to non-orientable laminations on orientable surfaces.

3.2. *Minimality.* We first recall the notion of connection and then prove that involutions without connections are essentially always minimal.

Definition 3.5. (Connection) A *connection* of a linear involution T is a triple (x, y, n) where x is a singularity of T^{-1}, y is a singularity of T, $n \geq 0$ and $T^n x = y$.

Let T be a linear involution without connections. Let

$$O = \bigcup_{n \geq 0} T^{-n}(\Sigma) \quad \text{and} \quad \hat{O} = O \cup \sigma_2(O) \tag{3.1}$$

be the negative orbit of the singular points and its closure under mirror image respectively. Then T is a bijection from $\hat{I} \backslash \hat{O}$ onto itself. Indeed, assume that $T(z) \in \hat{O}$. If $T(z) \in O$, then $z \in \hat{O}$. Next, if $T(z) \in \sigma_2(O)$, then $T(z) \in \sigma_2(T^{-n}(\Sigma)) = T^n(\sigma_2(\Sigma))$ for some $n \geq 0$. We cannot have $n = 0$ since $\sigma_2(\Sigma)$ is not in the image of T. Thus, $z \in T^{n-1}(\sigma_2(\Sigma)) = \sigma_2(T^{-n+1}(\Sigma)) \subset \sigma_2(O)$. Therefore, in both cases, $z \in \hat{O}$. The converse implication is proved in the same way. Note that $\hat{I} \backslash \hat{O}$ is dense in $\hat{I}$, and the non-negative orbit of any point of $\hat{I} \backslash \hat{O}$ is well defined.

Definition 3.6. (Minimality) A linear involution T on I without connections is minimal if for any point $z \in \hat{I} \backslash \hat{O}$ the non-negative orbit of z is dense in $\hat{I}$.

Note that, when a linear involution is orientable, that is, when it is a pair of interval exchange transformations (with or without flips), the interval exchange transformations can be minimal although the linear involution is not, since each component of $\hat{I}$ is stable by the action of T. Moreover, it is shown in [**13**] that non-coherent linear involutions are almost surely not minimal.

Let $X \subset I \times \{0, 1\}$. The *return time* ρ_X to X is the function from $I \times \{0, 1\}$ to $\mathbb{N} \cup \{\infty\}$ defined on X by

$$\rho_X(x) = \inf\{n \geq 1 \mid T^n(x) \in X\}. \tag{3.2}$$

The following result is proved in [**8**, Proposition 4.2] for the class of coherent involutions. The proof uses Keane's theorem and verifies that an interval exchange transformation without connections is minimal [**22**]. The proof of Keane's theorem also implies that, for each interval of positive length, the return time to this interval is bounded.

PROPOSITION 3.7. *Let T be a linear involution without connections on I. If T is non-orientable, it is minimal. Otherwise, its restriction to each component of $\hat{I}$ is minimal. Moreover, in both cases, for each interval of positive length included in $\hat{I}$, the return time to this interval takes a finite number of values.*

Proof. Consider the set $\widetilde{I} = \hat{I} \times \{0, 1\} = I \times \{0, 1\}^2$ and the transformation $\widetilde{T}$ on $\widetilde{I}$ defined for $(x, \delta) \in \widetilde{I}$ by

$$\widetilde{T}(x, \delta) = \begin{cases} (T(x), \delta) & \text{if } T \text{ is a translation on a neighborhood of } x, \\ (T(x), 1 - \delta) & \text{otherwise.} \end{cases}$$

Let T' be the transformation induced by $\widetilde{T}$ on $I' = I \times \{0, 0\}$. Note that, if $x \in I'$ is recurrent, that is $\widetilde{T}^n(x) \in I'$ for some $n > 0$, then the restriction of T' to some neighborhood of x is a translation. Indeed, there are an even number of indices i with $0 \le i < n$ such that T is a symmetry on a neighborhood of $T^i(x)$.

Let us show that T' is an interval exchange transformation. Let Σ be the set of singularities of T. For each $z \in \Sigma$, let $s(z)$ be the minimal integer $s > 0$ (or ∞) such that $\widetilde{T}^{-s}(z) \in I'$. Let $N = \{\widetilde{T}^{-s(z)}(z) \mid z \in \Sigma \text{ with } s(z) < \infty\}$. The set N divides I' into a finite number of disjoint open intervals. If J is such an open interval, it contains, by the Poincaré Recurrence theorem, at least one recurrent point $x \in I'$ for $\widetilde{T}$, such that $\widetilde{T}^n(x) \in I'$ for some $n > 0$. By definition of N, all the points of J are recurrent. Moreover, as we have seen above, the restriction of T' to J is a translation. This shows that T' is an interval exchange transformation.

We can now conclude the proof. Since T has no connection, T' has no connection. Thus, by Keane's theorem, it is minimal. This shows that the intersection with $I \times \{0\}$ of the non-negative orbit of any point in $I \times \{0\}$ is dense in $I \times \{0\}$. A similar proof shows that the same is true for $I \times \{1\}$. If T is non-orientable, the non-negative orbit of any $x \in I \times \{0\}$ contains a point in $I \times \{1\}$. Thus, its non-negative orbit is dense in $\hat{I}$. The same holds symmetrically for $x \in I \times \{1\}$.

Let J be an interval of positive length included in I. By Keane's theorem, the return time to $J \times \{0, 0\}$ relative to T' takes a finite number of values. Thus, the return time to $J \times \{0\}$ with respect to T also takes a finite number of values. A similar argument holds for an interval included in $I \times \{1\}$. □

4. *Measured foliations and linear involutions*

Now, let us introduce a geometric and topological viewpoint on natural involutions. The main features are measured foliations of surfaces introduced by Thurston (see [**18**] for an introduction, and see also [**21**]). They can be considered as two-dimensional extensions of linear involutions. They are defined on a compact surface X from which a finite number of points $\Sigma \subset X$ are removed. Poincaré sections of these measured foliations are then linear involutions.

A foliation is a decomposition of a surface as a union of leaves which are 1-dimensional. As an example, the plane $\mathbb{R}^2$ decomposes as a union of vertical lines. Let X be a (non-necessarily orientable) surface. A *foliation* on X is a covering of X by charts $\phi_i : X_i \to \mathbb{R}^2$ such that the transitions $\phi_i \circ \phi_j^{-1} : \phi_j(X_i \cap X_j) \to \phi_i(X_i \cap X_j)$ preserve vertical lines. In other words, they are of the form

$$\phi_i \circ \phi_j^{-1}(x, y) = (f_{ij}(x), g_{ij}(x, y)),$$

with $f_{ij}(x) = \pm x + c_{ij}$. In the chart ϕ_j, each stripe $x = a$ matches up with the stripe $x = f_{ij}(a)$ in X_i. Gluing all these stripes together, we obtain a *leaf* of the foliation which is a one-dimensional manifold immersed in X. Each leaf is hence homeomorphic to the circle $\mathbb{R}/\mathbb{Z}$ or the line $\mathbb{R}$. The surface X decomposes as the union of these leaves.

Given a non-singular smooth vector field, or more generally a line field, the integral curves of this field provide a foliation.

Example 4.1. Let T be the coherent linear involution on $I =]0, 1[$ represented in Figure 4.1. We choose $(3 - \sqrt{5})/2$ for the length of the interval I_c (or I_b). With this choice, T has no connection.

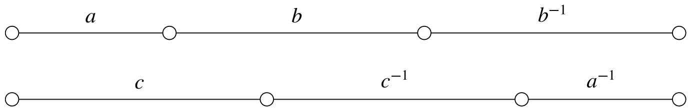

FIGURE 4.1. The 3-linear involution of Example 4.1.

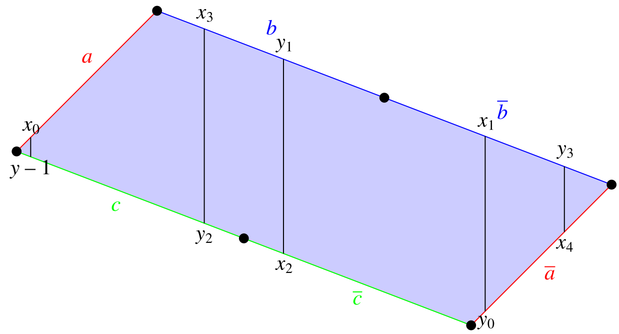

FIGURE 4.2. The foliation is made of vertical lines. The cutting sequence following a leaf is given by the iteration of the linear involution T (the notation follows the convention $\sigma_1(y_i) = x_i$ and $\sigma_2(y_i) = x_{i+1}$).

In Figure 4.2 we show an example of a foliation of a surface related to this linear involution. This surface is built from a polygon from which vertices are removed and where edges are glued with orientation-preserving isometry.

Now let X be a compact surface. A *singular foliation* on X is a foliation $\mathcal{F}$ defined on $X \setminus \Sigma$ where $\Sigma \subset X$ is a finite set of points such that in the neighborhood of each point of Σ the foliation is homeomorphic to the foliation of the punctured disc in $\mathbb{C}$ given by the line field $z^p(dz)^2 = I$: in other words, the leaves are the branches of $\gamma_c(t) = (It + c)^{1/(p/2+1)}$, where $c \in \mathbb{C}$ is a constant (see also Figure 4.3 for a picture). In this foliation, there are $p + 2$ singular leaves (which are half-lines that hit 0) that we call *separatrices*. We say that the singularity of the foliation has *angle* $(p + 2)\pi$ or *degree* p.

On the surface obtained from the polygon of Figure 4.2, one can check that the foliation has 4 singularities of degree $p = -1$ (or angle π).

A *transverse measure* on $\mathcal{F}$ is a measure μ defined on transverse arcs to $\mathcal{F}$ that is invariant under homotopy along the leaves and which is finite on compact intervals. A *measured foliation* is a singular foliation endowed with a transverse measure. We will see that linear involutions and measured foliations are essentially the same objects. In Figure 4.2, the natural transverse measure is simply the integral of dx along curves (where x is the natural horizontal coordinate in the plane).

A measured foliation is denoted by $(X, \Sigma, \mathcal{F}, \mu)$ or $(\mathcal{F}, \mu)$ when the space X and the set Σ are understood.

A *connection* of $\mathcal{F}$ is a finite leaf that joins two points of Σ.

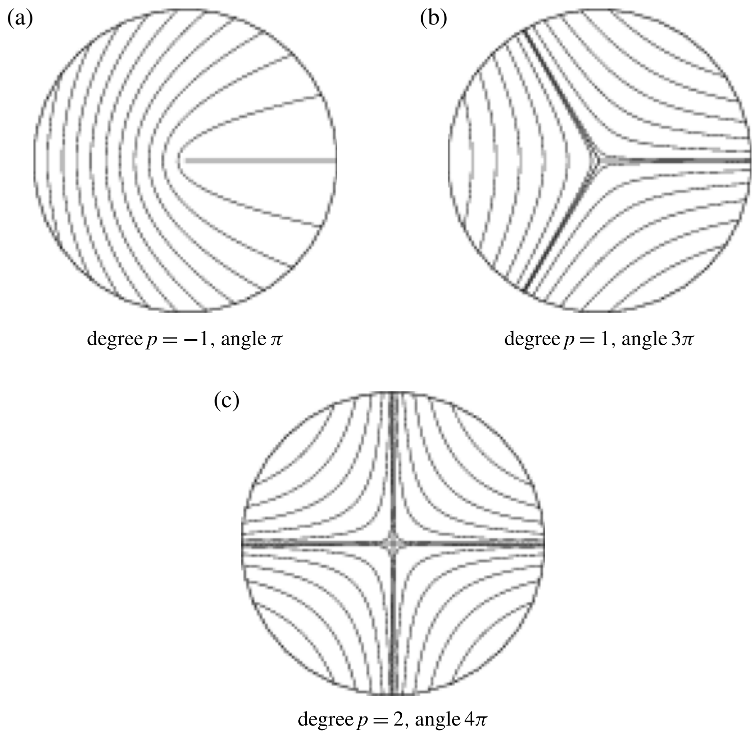

FIGURE 4.3. Chart around points of Σ.

Definition 4.2. Let $(X, \Sigma, \mathcal{F}, \mu)$ be a measured foliation without connections. A closed segment $I \subset X$ is admissible if:

- it is transverse to $\mathcal{F}$;
- its interior avoids Σ and both endpoints are on singular leaves; and
- the leaf segments that join one endpoint to a singularity do not intersect the interior of I.

We consider admissible intervals as being oriented, that is, having a start and an end. Because of the transverse measure, there is always a preferred parametrization for segments: we always assume that parametrization of a segment $\gamma : [0, t] \to X$ is such that $\mu(\gamma([s, s'])) = s' - s$. In other words, there is a unique parametrization such that $\mu|_I$ is the image of the Lebesgue measure. For a transverse segment I and $\delta > 0$ small enough, there is a neighborhood of I which is isomorphic to $[0, \mu(I)] \times [-\delta, \delta]$ and for which the leaves of the foliation on the rectangle are the vertical segments. For a piece of leaf that crosses the segment I, it hence makes sense to say *going up* or *going down*.

We define the *Poincaré map* of the foliation on $I \times \{0, 1\}$ as follows. For a point $x \in I$, we define $\sigma_1(x, 0)$ as the point $(y, i) \in I \times \{0, 1\}$ where y is the first point of the interior of I that is crossed by following the leaf from x and going up. If we arrive from above we set $i = 0$, and if not we set $i = 1$. Next, $\sigma_1(x, 1)$ is defined similarly, but following the leaf from x by going down. The map σ_1 is not defined if the leaf encounters a singularity before returning into I. The map σ_2 is the exchange $(x, 0) \mapsto (x, 1)$ and $(x, 1) \mapsto (x, 0)$. The transformation T is the composition $\sigma_2 \circ \sigma_1$. The sequence $(x, 0), T(x, 0), T^2(x, 0), \ldots$ is by construction the sequence of intersections of the leaf from x with I.

Remark 4.3. The notion of mixed return words is explained by the way the Poincaré map of the foliation works (see Definition 5.10 below).

The *total angle* of a foliation is the sum of the angles of the singularities.

LEMMA 4.4. *Let $(X, \Sigma, \mathcal{F}, \mu)$ be a measured foliation without connections of total angle $(2k-2)\pi$. Let I be an admissible interval. Then the Poincaré map induced on $I \times \{0, 1\}$ is a k-linear involution without connections.*

Proof. If the foliation has no connection, then each infinite half-leaf intersects I. We consider singularities for the Poincaré map, in other words the points in I that run into a singularity before going back to I. This set cuts the domain $I \times \{0, 1\}$ into subintervals. As the transverse measure is preserved, the Poincaré map is an isometry restricted to each of these subintervals. Hence, it is a linear involution.

For each subinterval I_a, let R_a be the rectangle made by the union of the leaf segments that start from I_a to $T(I_a)$. On each of the two vertical boundaries of these rectangles there is exactly one singularity, except for two rectangles among the extreme rectangles (there are two, three or four such rectangles). It follows that there are k pairs of subintervals for the Poincaré map. □

Note that if $p_1, \ldots, p_s$ are the degrees of the singularities, then the sum of the angles is $(p_1+2)\pi + \cdots + (p_s+2)\pi = (2k-2)\pi$, and thus

$$p_1 + \cdots + p_s + 2s + 2 = 2k.$$

In the example of Figure 4.2, one has $s = 4$, $p_1 = p_2 = p_3 = p_4 = -1$ and $k = 3$.

The following lemma is the converse of Lemma 4.4.

LEMMA 4.5. *Let T be a linear involution without connections. Then there exists a measured foliation $(X, \Sigma, \mathcal{F}, \mu)$ without connections and an admissible interval $I \subset X$ such that T is conjugate to the Poincaré map of the foliation $\mathcal{F}$ on I.*

Proof. We just use the reverse procedure as in the proof of Lemma 4.4. For each subinterval I_a, we consider a rectangle $R_a = I_a \times [0, 1]$. The vertical boundaries of the rectangles can be glued together to give a foliation. Note that there is no need to glue the vertical sides of the rectangles by isometry since we are only interested in the transverse measure dx. □

The pair $(\mathcal{F}, \mu, I)$ of a measured foliation and an admissible interval associated with T as above is called a *suspension* of T.

5. *Natural codings*

We now focus on return words of linear foliations. Algebraic information on the set of return words (see Theorem 6.4 below) will follow from the remark that a section captures the geometry of the surface (see Lemma 6.1) and that the free group is geometrically seen as the fundamental group $\pi_1(X \backslash \Sigma)$.

5.1. *Natural codings of linear involutions.* In this section, we introduce the natural coding of a linear involution T. It is obtained by first coding the orbits under T with respect to the partition provided by the intervals I_a ($a \in A \cup A^{-1}$), and then by taking the language of the associated symbolic dynamical system.

Let T be a linear involution on I, let $\hat{I} = I \times \{0, 1\}$ and let $\hat{O}$ be the set defined by equation (3.1). Given $z \in \hat{I} \setminus \hat{O}$, the *infinite natural coding* of T relative to z is the infinite word $\Sigma_T(z) = a_0 a_1 \cdots$ on the alphabet $A \cup A^{-1}$ defined by

$$a_n = a \quad \text{if } T^n(z) \in I_a.$$

First, we observe that the infinite word $\Sigma_T(z)$ is reduced. Indeed, assume that $a_n = a$ and $a_{n+1} = a^{-1}$ with $a \in A \cup A^{-1}$. Set $x = T^n(z)$ and $y = T(x) = T^{n+1}(z)$. Then $x \in I_a$ and $y \in I_{a^{-1}}$. But, $y = \sigma_2(u)$ with $u = \sigma_1(x)$. Since $x \in I_a$, we have $u \in I_{a^{-1}}$. This implies that $y = \sigma_2(u)$ and u belong to the same component of $\hat{I}$, which is a contradiction.

Definition 5.1. (Natural coding) Let T be a linear involution. We let $\mathcal{L}(T)$ denote the set of factors of the infinite natural codings of T. We say that $\mathcal{L}(T)$ is the *natural coding* of T.

As classically done in symbolic dynamics for codings, the set $\mathcal{L}(T)$ can easily be described in terms of intervals associated with factors, obtained by refining the coding partition.

LEMMA 5.2. *Let T be a linear involution. For a non-empty word $u = a_0 a_1 \cdots a_{m-1}$ on $A \cup A^{-1}$, we define*

$$I_u = I_{a_0} \cap T^{-1}(I_{a_1}) \cap \cdots \cap T^{-m+1}(I_{a_{m-1}}).$$

By convention, $I_\varepsilon = \hat{I}$. We have

$$u \in \mathcal{L}(T) \Longleftrightarrow I_u \neq \emptyset.$$

Proof. Each set I_u is a (possibly empty) open interval. Indeed, this is true if u is a letter. Next, assume that I_u is an open interval. Note that

$$I_{au} = I_a \cap T^{-1}(I_u).$$

Then, for $a \in A \cup A^{-1}$, we have $T(I_{au}) = T(I_a) \cap I_u$ and thus $T(I_{au})$ is an open interval. Since $I_{au} \subset I_a$, $T(I_{au})$ is the image of I_{au} by a continuous map and thus I_{au} is also an open interval.

Let u be a factor of $\mathcal{L}(T)$. By Definition 5.1, there exists a point $z \in \hat{I} \setminus \hat{O}$ such that u is a factor of $\Sigma_T(z)$. In particular, $T^{-n}(z)$ never belongs to the set Σ of singular points, for any non-negative n. Hence, for any $z \in \hat{I} \setminus \hat{O}$, one has $z \in I_u$ if and only if u is a prefix of $\Sigma_T(z)$.

If u is a factor of $\Sigma_T(z)$ for some $z \in \hat{I} \setminus \hat{O}$, then $T^n(z) \in I_u$ for some $n \geq 0$ and thus $I_u \neq \emptyset$. Conversely, if $I_u \neq \emptyset$, since I_u is an open interval, it contains some $z \in \hat{I} \setminus \hat{O}$. Then u is a prefix of $\Sigma_T(z)$ and thus $u \in \mathcal{L}(T)$. □

Observe that if T is non-orientable and without connections, then by Proposition 3.7, $\mathcal{L}(T)$ is the set of factors of $\Sigma_T(z)$ for any $z \in \hat{I} \setminus \hat{O}$, and so the set of factors of $\Sigma_T(z)$ does not depend on z. Indeed, if $I_u \neq \emptyset$, since the orbit of z is dense in $\hat{I}$, there is an $n \geq 0$ such that $T^n(z) \in I_u$ and thus u is a factor of $\Sigma_T(z)$.

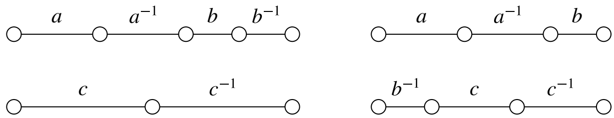

FIGURE 5.1. The transforms T' and T'' of T by Rauzy induction.

PROPOSITION 5.3. *Let $T = \sigma_2 \circ \sigma_1$ be a linear involution. For any non-empty word $u \in \mathcal{L}(T)$, one has $I_{u^{-1}} = \sigma_1 T^{|u|-1}(I_u)$. Consequently, the set $\mathcal{L}(T)$ is closed by taking inverses. It is thus a laminary set.*

Proof. To prove the assertion, we use an induction on the length of u. The property holds for $|u| = 1$ by definition of σ_1. Next, consider $u \in \mathcal{L}(T)$ and $a \in A \cup A^{-1}$ such that $ua \in \mathcal{L}(T)$. We assume by induction that $I_{u^{-1}} = \sigma_1 T^{|u|-1}(I_u)$.

Since $T^{-1} = \sigma_1 \circ \sigma_2$,

$$\begin{aligned}\sigma_1 T^{|u|}(I_{ua}) &= \sigma_1 T^{|u|}(I_u \cap T^{-|u|}(I_a)) = \sigma_1 T^{|u|}(I_u) \cap \sigma_1(I_a)\\ &= \sigma_1\sigma_2\sigma_1 T^{|u|-1}(I_u) \cap \sigma_1(I_a) = \sigma_1\sigma_2(I_{u^{-1}}) \cap I_{a^{-1}} = I_{a^{-1}u^{-1}},\end{aligned}$$

where the last equality results from $I_{a^{-1}u^{-1}} = T^{-1} I_{u^{-1}} \cap I_{a^{-1}}$.

We easily deduce that the set $\mathcal{L}(T)$ is closed by taking inverses. Furthermore, it is a factorial subset of the free group F_A. It is thus a laminary set. □

Example 5.4. Let T be the linear involution of Example 4.1. This linear involution is one of the simplest non-trivial examples of linear involution we can think of, with respect to the size of the alphabet. By non-trivial, we mean, in particular, that it is without connections and non-orientable (it thus admits odd and even letters, see Examples 5.8 and 6.11 below). This simple example illustrates how Rauzy induction can be used to understand natural codings. Recall that Rauzy induction was initially defined for interval exchanges in [**26**] and extended to linear involutions in [**8**]. Recall also that induction is intimately connected with return words.

The set $S = \mathcal{L}(T)$ can actually be defined directly as the set of factors of the substitution

$$f : a \mapsto cb^{-1}, \quad b \mapsto c, \quad c \mapsto ab^{-1},$$

which extends to an automorphism of the free group F_A. Thus, the natural coding of T bears some analogy with the language of the Fibonacci morphism $a \mapsto ab,\ b \mapsto a$ (see also Example 6.8).

Indeed, the Rauzy induction applied to T gives the linear involution T' represented in Figure 5.1 on the left. It is the transformation induced by T on the interval obtained by erasing the smallest interval on the right, namely $I_{a^{-1}}$.

The Rauzy induction applied on T' is obtained by erasing the smallest interval on the right, namely $I_{b^{-1}}$. It gives a transformation T'' represented on the right of Figure 5.1 .

The transformation T'' is the same as T up to normalization of the length of the interval, exchange of the two components and the permutation (written in cycle form) $\pi = (a\ c\ b\ a^{-1}\ c^{-1}\ b^{-1})$ (see Figure 5.1) which sends a to c, c to b and so on.

Set $S=\mathcal{L}(T)$, $S'=\mathcal{L}(T')$ and $S''=\mathcal{L}(T'')$. Since T' is obtained from T by a Rauzy induction, there is an associated automorphism τ' of the free group such that $S=\mathrm{Fact}(\tau'(S'))$ (where the notation Fact( ) stands for the set of factors). One has actually $\tau' : a\mapsto ab^{-1}, b\mapsto b, c\mapsto c$. Similarly, one has $S'=\mathrm{Fact}(\tau''(S''))$ with $\tau'' : a\mapsto a, b\mapsto bc^{-1}, c\mapsto c$. Furthermore, $S''=\pi^{-1}(S)$. Set $\tau=\tau'\circ\tau''$. It is easy to verify that $f=\tau\circ\pi^{-1}$. Since $S=\mathrm{Fact}(\tau(S''))=\mathrm{Fact}(\tau\pi^{-1}(S))=\mathrm{Fact}(f(S))$, we obtain that S is the set of factors of the fixpoint of f as claimed above.

5.2. *Orientability and uniform recurrence.* Here, we gather basic properties of the language $\mathcal{L}(T)$ of a linear involution. We recall that the notion of orientability for a laminary set was introduced in §2.

PROPOSITION 5.5. *Let T be a linear involution. If T is orientable, then $\mathcal{L}(T)$ is orientable. The converse is true if T is without connections.*

Proof. Let T be a linear involution and let $S=\mathcal{L}(T)$. Assume that T is orientable. Set $S_+=\{u\in S \mid I_u\subset I\times\{0\}\}\cup\{\varepsilon\}$ and $S_-=\{u\in S\mid I_u\subset I\times\{1\}\}\cup\{\varepsilon\}$. Then $S=S_+\cup S_-$. Since T is orientable, we have $u\in S_+$ (respectively $u\in S_-$) if and only if all letters of u are in S_+ (respectively in S_-). This shows that $S_+\cap S_-=\{\varepsilon\}$, that S_+, S_- are factorial, and that $u\in S_+$ if and only if $u^{-1}\in S_-$. Thus S is orientable.

Conversely, assume that T is non-orientable and is without connections. Let $a\in A$ be such that $I_a, I_{a^{-1}}\subset I\times\{0\}$. Since T is minimal by Proposition 3.7, there is some $z\in I_a$ and $n>0$ such that $T^n(z)\in I_{a^{-1}}$. Thus, S contains a word of the form aua^{-1}. This implies that S is non-orientable. □

The following statement can easily be deduced from the similar statement for interval exchange transformations (see [**7**, p. 392]).

PROPOSITION 5.6. *Let T be a linear involution without connections. If T is non-orientable, then $\mathcal{L}(T)$ is uniformly recurrent. Otherwise, $\mathcal{L}(T)$ is uniformly semi-recurrent.*

Proof. Set $S=\mathcal{L}(T)$. Let $u\in S$ and let N be the maximal return time to I_u (this exists by Proposition 3.7). Thus, for any $z\in\hat{I}$ such that the return time $\rho_{I_u}(z)$ is finite (see equation (3.2)), we have $\rho_{I_u}(z)\le N$. Let w be a word of S of length $N+|u|$ and let $z\in\hat{I}\setminus\hat{O}$ be such that $\Sigma_T(z)$ begins with w.

If T is non-orientable, by Proposition 3.7, it is minimal. Thus, there exists $n>0$ such that $T^n(z)\in I_u$. This implies that $\rho_{I_u}(z)$ is finite and thus that $\rho_{I_u}(z)\le N$. This implies in turn that u is a factor of w. We conclude that S is uniformly recurrent.

If T is orientable, then the restriction of T to each component of $\hat{I}$ is minimal. By Proposition 5.5, S is orientable. Thus, I_u and $I_{u^{-1}}$ cannot be included in the same component of $\hat{I}$, since otherwise S would contain a word of the form uvu^{-1}, and S would be non-orientable. Thus, I_w is in the same component as I_u or $I_{u^{-1}}$, and we conclude as above that u or u^{-1} is a factor of w. This shows that S is uniformly semi-recurrent. □

5.3. *Return words and the even group.* In this section, we first introduce odd and even words, and then discuss various variations on the notion of return words.

Definition 5.7. (Even group) Let T be a linear involution without connections.

We say that a letter $a \in A$ is *even* (with respect to T) if I_a and $I_{a^{-1}}$ belong to distinct components of $\hat{I}$ and *odd*, otherwise.

A reduced word is said to be *even* if it has an even number of odd letters and said to be *odd*, otherwise. In particular, if T is orientable, all words are even.

The *even group* is the subgroup of the free group F_A formed by the even words.

Note that a word w is even if and only if, for any $z \in I_w$, the points z and $T^{|w|}(z)$ belong to the same component. Since $\sigma_2 I_{w^{-1}} = T^{|w|}(I_w)$ according to Proposition 5.3, w is even if and only if I_w and $I_{w^{-1}}$ belong to distinct components of $\hat{I}$. Hence, a word w is even if and only if I_w and $T^{-|w|}I_w$ belong to the same component.

If T is assumed to be non-orientable, the even group is a subgroup of index 2 of F_A; it thus has rank two Card $A - 1$ according to Schreier's formula.

Example 5.8. Let T be the linear involution of Example 4.1. The letter a is even and the letters b, c are odd. The even group is generated by the set $Z = \{a, b\bar{a}c, b\bar{c}, \bar{b}\bar{c}, \bar{b}c\}$.

We now introduce several definitions of return words. Let T be a linear involution relative to the alphabet A and let $S = \mathcal{L}(T)$ be its natural coding. Recall that S is a factorial subset of the free group F_A.

For a set $W \subset S$, a *complete return word* to W is a word of S which has a proper prefix in W and a proper suffix in W, and that has no internal factor in W. If S is uniformly recurrent (in particular, if T is non-orientable and without connections, by Proposition 5.6), the set of complete return words to W is finite for any finite set W.

We now focus on return words for two types of sets W, namely sets reduced to one word or symmetric sets of the form $\{w, w^{-1}\}$.

By considering the set $\{w\}$, one arrives at the classical notion of return words. For any $w \in S$, a *right return word* to w in S is a word u such that wu is a complete return word to $\{w\}$. We denote by $\mathcal{R}_S(w)$ the set of right return words to w in S. We define left return words similarly.

Remark 5.9. Note that all elements of $\mathcal{R}_S(x)$ are even. Indeed, if $w \in \mathcal{R}_S(x)$, we have $xw = vx$ for some $v \in S$. We assume, without loss of generality, that x is odd and that $I_x \subset I \times \{0\}$. Take $z \in I_{xw}$. Then $T^{|x|}(z) \in I \times \{1\}$ since x is odd. One has $T^{|x|}(z) \in I_w$. Hence, $I_w \subset I \times \{1\}$. But $T^{|w|}(I_w) \subset T^{-|x|}I_x \subset I \times \{1\}$ (again since x is odd). Hence, $T^{|w|}(I_w)$ and I_w belong to the same component and w is even. The other cases can be handled similarly.

For $w \in S$, we also consider complete return words to the set $W = \{w, w^{-1}\}$ in S. We let $\mathcal{CR}_S(w)$ denote this set and call its elements the *complete return words to* $\{w, w^{-1}\}$.

In order to provide a connection between return words and elements of a symmetric basis of the free group, we need to introduce a further definition, which is the analog of return words in symbolic dynamics.

Definition 5.10. (Mixed return words) With a complete return word u to the set $\{w, w^{-1}\}$, we associate a word $N(u)$ as follows: if u has w as prefix, we erase it and if u has a suffix w^{-1} we also erase it. We let $\mathcal{MR}_S(w)$ denote the set of mixed return words.

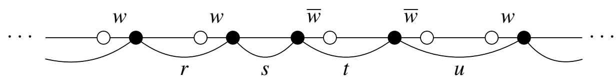

FIGURE 5.2. A uniformly recurrent infinite word factorized as an infinite product $\cdots rstu \cdots$ of mixed return words to w.

The convention chosen for the transformation N corresponds to the induction on $I_{w^{-1}} \cup \sigma_2(I_{w^{-1}})$ (see Lemma 5.12 below).

Note that the two operations described above can be made in any order since w and w^{-1} cannot overlap. Note also that $\mathcal{MR}_S(w)$ is symmetric and that $w^{-1}\mathcal{MR}_S(w)w = \mathcal{MR}_S(w^{-1})$.

If T is orientable, then $\mathcal{MR}_S(w)$ is equal to the union of the set of right return words to w with the set of left return words to w^{-1}.

Observe that any uniformly recurrent biinfinite word x whose set of factors is S can be uniquely written as a concatenation of mixed return words (see Figure 5.2). Note also that successive occurrences of w may overlap but that successive occurrences of w and w^{-1} cannot.

Example 5.11. Let T be the linear involution of Example 4.1. We have

$$\begin{aligned}
\mathcal{CR}_S(a) &= \{a\bar{b}cb\bar{a},\ a\bar{b}cb\bar{c}a,\ \bar{a}c\bar{b}\bar{c}a,\ a\bar{b}\bar{c}b\bar{a},\ \bar{a}cb\bar{c}a,\ \bar{a}c\bar{b}\bar{c}b\bar{a}\},\\
\mathcal{CR}_S(b) &= \{b\bar{a}cb,\ b\bar{a}c\bar{b},\ b\bar{c}a\bar{b},\ \bar{b}cb,\ \bar{b}\bar{c}a\bar{b},\ \bar{b}\bar{c}b\},\\
\mathcal{CR}_S(c) &= \{cb\bar{a}c,\ cb\bar{c},\ c\bar{b}\bar{c},\ \bar{c}a\bar{b}c,\ \bar{c}a\bar{b}\bar{c},\ \bar{c}b\bar{a}c\},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{MR}_S(a) &= \{\bar{b}cb,\ \bar{b}cb\bar{c}a,\ \bar{a}c\bar{b}\bar{c}a,\ \bar{b}\bar{c}b,\ \bar{a}cb\bar{c}a,\ \bar{a}c\bar{b}\bar{c}b\},\\
\mathcal{MR}_S(b) &= \{\bar{a}cb,\ \bar{a}c,\ \bar{c}a,\ \bar{b}cb,\ \bar{b}\bar{c}a,\ \bar{b}\bar{c}b\},\\
\mathcal{MR}_S(c) &= \{b\bar{a}c,\ b,\ \bar{b},\ \bar{c}a\bar{b}c,\ \bar{c}a\bar{b},\ \bar{c}b\bar{a}c\}.
\end{aligned}$$

The reason for introducing the notion of mixed return words (see Definition 5.10) comes from the fact that we are interested in the transformation induced on $I_w \cup \sigma_2(I_w)$, according to §4 (see Remark 4.3). The natural coding of a point in I_w begins with w while the natural coding of a point z in $\sigma_2(I_w)$ is preceded by w^{-1} in the sense that the natural coding of $T^{-|w|}(z)$ begins with w^{-1}. To be more precise, the convention chosen for the transformation N corresponds to the induction on $I_{w^{-1}} \cup \sigma_2(I_{w^{-1}})$, such as shown by the following lemma. Recall that the notation ρ_X stands for the return time to X.

LEMMA 5.12. *Let T be a linear involution without connections and w a non-empty word in its natural coding $\mathcal{L}(T)$. Let $K_w = I_{w^{-1}} \cup \sigma_2(I_{w^{-1}})$. Then the set of mixed return words to w are exactly the prefixes of length $\rho_{K_w}(z)$ of the infinite natural coding of points $z \in K_w$.*

Proof. Let u be the prefix of length $\rho_{K_w}(z)$ of $\Sigma_T(z)$ for some $z \in K_w$. First, let us recall that $\sigma_2(I_{w^{-1}}) = T^{|w|}(I_w)$ (Proposition 5.3). Assume that the length of u is larger than or equal to the length of w. If $z \in I_{w^{-1}}$, then u starts with w^{-1} while, if $z \in \sigma_2(I_{w^{-1}})$, then wu

is in $\mathcal{L}(T)$. Similarly, if $T^{|u|}(z) \in I_{w^{-1}}$, then uw^{-1} is in $\mathcal{L}(T)$ while, if $T^{|u|}(z) \in \sigma_2(I_{w^{-1}})$, then u ends with w. In all four possible cases, u, wu, uw^{-1} and wuw^{-1} are in $\mathcal{L}(T)$.

Let

$$p = \begin{cases} \varepsilon & \text{if } z \in I_{w^{-1}}, \\ w & \text{if } z \in \sigma_2(I_{w^{-1}}), \end{cases} \qquad \text{and} \quad s = \begin{cases} w^{-1} & \text{if } T^{|u|}(z) \in I_{w^{-1}}, \\ \varepsilon & \text{if } T^{|u|}(z) \in \sigma_2(I_{w^{-1}}). \end{cases}$$

Since $I_{w^{-1}}$ and $\sigma_2(I_{w^{-1}})$ are included into two distinct components, there is no cancellation in the product pus. Moreover, $|pus| \geq |u|$ and hence pus starts and ends with an occurrence of w or w^{-1}. Thus, one has $N(pus) = u$ and pus is a complete return word to $\{w, w^{-1}\}$.

Conversely, let u be a mixed return word to w and let u' be the complete return word such that $u = N(u')$. Write $u' = pus$. First, assume that $u' = wu$. Then wu ends with w. For any point $y \in I_{u'}$, set $x = T^{|w|}(y)$. Then $x \in T^{|w|} I_w = \sigma_2(I_{w^{-1}})$, $x \in I_u$ and thus $T^{|u|}x \in \sigma_2(I_{w^{-1}})$ and $\rho_{K_w}(x) = |w|$. Hence, u is the prefix of length $\rho_{J_w}(x)$ of $\Sigma_T(x)$. The proof in the three other cases is similar. □

We end this section by introducing a further variation of return words, adapted to subgroups of the free group (this concept will be highlighted in Theorem 6.9 below).

Definition 5.13. (Prime words) Let G be a subgroup of the free group F_A. Let S be a laminary set. The *prime words* in S with respect to G are the non-empty words in $G \cap S$ without a proper non-empty prefix in $G \cap S$.

Example 5.14. Let T be the linear involution of Example 4.1. The set of prime words with respect to the even group is the set $Z \cup Z^{-1}$, where Z is as in Example 5.8.

6. *Return words and fundamental group*

We now interpret the definitions of 'return words' that we have seen so far (to a word, or with respect to a subgroup via the concept of prime words) in geometrical terms.

We consider a punctured surface (X, Σ). Fixing a base point x_0, recall that the *fundamental group* $\pi_1(X \setminus \Sigma, x_0)$ is the set of equivalence classes of loops in $X \setminus \Sigma$ based at x_0 up to homotopy. One ingredient of our main results (Theorems 6.4 and 6.9) is that each admissible interval for the foliation (in the sense of Definition 4.2) is associated with a symmetric basis of the fundamental group, as we shall see below. Furthermore, the fundamental group is a free group.

Let $(X, \Sigma, \mathcal{F}, \mu)$ be a measured foliation and assume that Σ is non-empty. Let I be an admissible interval and let x_0 be any point of I. By Lemma 4.4, the domain $I \times \{0, 1\}$ of the Poincaré map T is cut into $2k$ subintervals by the first return map. With each subinterval I_a we associate an element of $\pi_1(X \setminus \Sigma, x_0)$ as follows. Let x be a point in that subinterval, we consider the loop $\gamma(x)$ which is the concatenation of:

- the segment in I that joins x_0 to x;
- the piece of leaf that joins x to $x' = T(x)$; and
- the segment in I that joins x' to x_0.

The homotopy class of $\gamma(x)$ only depends on the subinterval to which x belongs. We let $\Gamma(X, I, x_0)$ denote the set of equivalence classes of loops in $\pi_1(X \setminus \Sigma, x_0)$ obtained by that process. The following lemma shows, in particular, that there are $2k$ classes.

LEMMA 6.1. *Let $(X, \Sigma, \mathcal{F}, \mu)$ be a measured foliation with total angle $(2k-2)\pi$. Then, if Σ is non-empty, the fundamental group of $X\backslash\Sigma$ is a free group on k generators. Moreover, for any admissible interval I in X and any $x_0 \in I$, the set $\Gamma(X, I, x_0)$ is a symmetric basis of $\pi_1(X\backslash\Sigma, x_0)$.*

Proof. Let I be an admissible interval. We consider the k loops obtained from the above construction. With an homotopy fixing x_0, one can easily realize the loops in such way that the only common point between any two is x_0. We let $Y \subset X\backslash\Sigma$ denote this set of d loops. Now we show that the punctured surface $X\backslash\Sigma$ is homotopic to Y. We may decompose the surface $X\backslash\Sigma$ into zippered rectangles as in Lemmas 4.4 and 4.5: we cut the surface along each singular leaf, from the singularities until the first time it hits the interior of I. In each rectangle there is exactly one loop passing through. It is easy to see that, by a continuous deformation, we can shrink each rectangle to that loop. In other words we build a homotopy to Y.

Now Y is a connected sum of k loops (also called a rose) and its fundamental group is a free group of rank k generated by each curve that goes once through a loop. □

6.1. *Return words and bases of the free group.* We now have gathered all the necessary information to deduce algebraic properties of mixed return words.

Let $T : I \times \{0, 1\} \to I \times \{0, 1\}$ be a linear involution relative to the alphabet A and let $S = \mathcal{L}(T)$. We have introduced with Definition 4.2 the notion of an admissible interval $I \subset X$ with respect to a measured foliation $(X, \Sigma, \mathcal{F}, \mu)$. We can directly formulate a similar definition for an open interval $J \subset I$ with respect to a linear involution T defined on I as follows.

Definition 6.2. (Admissible interval) Let T be a linear involution without connections defined on the interval I. The open interval $J =]u, v[$ with $J \subset I$ is *admissible* with respect to T if, for each of its two endpoints $x = u, v$:
(i) x is equal to the left or to the right boundary of I;
(ii) there is a singularity z of T^{-1} such that $x = T^n(z)$ and $T^k(z) \notin J$ for $0 \le k \le n$; or
(iii) there is a singularity z of T such that $z = T^n(x)$ and $T^k(x) \notin J$ for $0 \le k \le n$.

The term 'admissible' was originally introduced by Rauzy [**26**] for interval exchanges.

It is clear that if J is admissible with respect to T, then it is admissible with respect to any suspension $(\mathcal{F}, \mu, I)$ of T. Hence, for any admissible interval of I with respect to T, the transformation induced on I is a k-linear involution without connections, according to Lemma 4.4. Furthermore, for any admissible interval of I, the Poincaré map of the foliation is the Poincaré map of the linear involution on the union $I \cup \sigma_2(I)$.

The following result is proved in [**14**] for interval exchange transformations. The proof for linear involutions is the same. Recall that the intervals I_w, $w \in S$, are defined in §5.1.

PROPOSITION 6.3. *Let T be a k-linear involution without connections on I. The interval I_w, seen as a subinterval of I, is admissible with respect to T.*

Proof. Let T be a k-linear involution. Recall that Σ is the set of $2k-2$ division points separating the intervals I_a for $a \in A \cup A^{-1}$.

Let $n \geq 1$. Since T is without connections, $T^{-i}(z)$ is well defined for any $z \in \Sigma$ and for any i such that $0 \leq i \leq n-1$. Let $P_n = \{T^{-i}(z) \mid z \in \Sigma, 0 \leq i \leq n-1\} \cup (\{\lambda\} \times \{0, 1\})$, where λ stands for the left endpoint of the interval I. One has $\mathrm{Card}(P_n) = (2k-2)n + 2$. Consider two points z and z' in $\hat{I} \backslash \hat{O}$ that belong to two different intervals of the partition by open intervals of $I \times \{0, 1\}$ made by the points of P_n. Then the prefixes of size n of their respective infinite natural codings differ. On the other hand, the left boundary of each I_w, $|w| = n$, is the left boundary of some $T^{-i}(I_a)$ for some $0 \leq i \leq n-1$ and some $a \in A \cup A^{-1}$. This proves that P_n is the set of $2(k-1)n + 2$ left boundaries of the intervals I_w for all words w with $|w| = n$ and that the family $(I_w)_{|w|=n}$ forms a partition of $I \times \{0, 1\}$ (up to the points of P_n).

Let $I_w =]u, v[$ and $w = a_0 a_1 \cdots a_{n-1}$. We assume that $u \neq \lambda$. By construction, there exist a point $z \in \Sigma$ and an integer i with $0 \leq i \leq n-1$ such that $u = T^{-i}(z)$, where $I_{a_i} =]z, t[$ for some t in I or equal to the right boundary of I. For any k with $0 \leq k \leq n-1$, the point $T^{-k}(z)$ is the left boundary of some interval I_y, with $|y| = n$. Thus, in particular, one gets $T^k(u) \notin I_w$, for $0 \leq k \leq i$.

The same reasoning applies to the right boundary v of I_w. □

We now can state our main result concerning return words.

THEOREM 6.4. *Let S be the natural coding of a linear involution without connections on the alphabet A. For any $w \in S$, the set of mixed return words to w is a symmetric basis of the free group F_A.*

Proof of Theorem 6.4. Let T be a linear involution without connections relative to the alphabet A. By Lemma 4.5, there exist a measured foliation $(X, \Sigma, \mathcal{F}, \mu)$ and an admissible interval $I \subset X$ such that T is conjugate to the Poincaré map of $\mathcal{F}$ on I. Let w be a non-empty word of the natural coding $S = \mathcal{L}(T)$. By Proposition 6.3, the subinterval I_w is admissible for the linear involution T. Let x_0 be a point in I_w. Recall F_A stands for the free group on the alphabet A. We have a natural identification $F_A \to \pi_1(X \backslash \Sigma, x_0)$ given by Lemma 6.1. Since I_w is admissible, using Lemma 5.12, the same construction provides an identification of the subgroup $\Gamma(X, I_w, x_0)$ generated by the mixed return words and $\pi_1(X \backslash \Sigma, x_0)$. This shows that the set of mixed return words is a symmetric basis of F_A. □

Theorem 6.4 thus provides bases of the free group within a given natural coding by taking mixed return words with respect to a given factor w.

Example 6.5. The set of $\mathcal{MR}_S(c)$ in Example 5.11 provides a symmetric basis of the free group, whereas $\mathcal{CR}_S(c)$ is not a symmetric basis of the free group.

One also deduces the following cardinality result, which is the counterpart for linear involutions of [**6**, Theorem 3.6], that holds for tree sets, by noticing that the set of mixed return words $\mathcal{MR}_S(w)$ has the same cardinality as the set of complete return words $\mathcal{CR}_S(w)$.

COROLLARY 6.6. *Let T be a linear involution without connections relative to the alphabet A. For any $w \in \mathcal{L}(T)$, the set of complete return words to $\{w, w^{-1}\}$ has $2\,\mathrm{Card}(A)$ elements.*

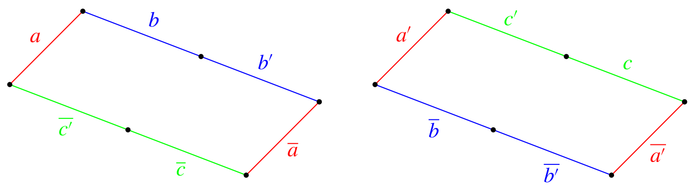

FIGURE 6.1. The orientation covering of the suspension of Figure 4.2. The choice of letters is made in order that only positive letters or negative letters appear in the coding of an orbit.

6.2. *Prime words and coverings.* We now prove an analogue of Theorem 6.4 for prime words with respect to a subgroup of the free group. This will be Theorem 6.9 below. We will first consider surface coverings that correspond to subgroups of $\pi_1(X\backslash\Sigma)$. From this correspondence, we will obtain a proof of Theorem 6.9.

First, let us quickly recall the Galois correspondence of coverings. Let X be a compact connected surface and Σ a finite set of points. A *covering* of X of degree d is a compact connected surface Y with a continuous map $f : Y \to X$ such that for each $x \in X\backslash\Sigma$ there exists a connected neighborhood U of x such that $f^{-1}(U)$ is a disjoint union of d open sets $f^{-1}(U) = U_1 \cup U_2 \cup \cdots \cup U_d$ and for each $i \in \{1, \ldots, d\}$, $f : U_i \to U$ is a homeomorphism. In our case, we consider more generally a *ramified covering* with ramifications contained in Σ. For points $x \in X\backslash\Sigma$ we keep the same condition, but for points $x \in \Sigma$ we allow the pre-image to be a union of $m \leq d$ open sets $U_1 \cup U_2 \cup \cdots U_m$ such that f restricted to U_i is of the form $z \mapsto z^{p_i}$ for some $p_i \geq 0$ from the unit disc in $\mathbb{C}$ to itself. One can show that $p_1 + p_2 + \cdots + p_m = d$. In other words, the degree is constant if we count multiplicities.

Two coverings $f : Y \to X$ and $f' : Y' \to X$ are *equivalent* if there exists an homeomorphism $g : Y \to Y'$ such that $f = f' \circ g$.

If γ is a loop in Y, then $f(\gamma)$ is a loop in X. Hence, for any $y_0 \in Y$ we get a map $f_* : \pi_1(Y\backslash f^{-1}(\Sigma), y_0) \to \pi_1(X\backslash\Sigma, f(y_0))$. The map f_* is injective and its image is of finite index in $\pi_1(X\backslash\Sigma, f(y_0))$.

The following result establishes a Galois correspondence between coverings of finite degree of X ramified over Σ and subgroups of $\pi_1(X\backslash\Sigma)$. For a proof, see [**21**] or [**19**].

THEOREM 6.7. *Let X be a compact connected surface and let $\Sigma \subset X$ be a finite set. Let Y be a covering of X of degree d. Then, the map $(f : Y \to X) \mapsto f_*(\pi_1(Y\backslash f^{-1}(\Sigma)))$ induces a bijection between equivalence classes of coverings of degree d ramified over Σ and conjugacy classes of subgroups of $\pi_1(X\backslash\Sigma)$ of index d.*

Example 6.8. Let T be the linear involution of Example 4.1. It is without connections and non-orientable,and thus the group of even words is a subgroup of index $d = 2$. The covering of degree two of its suspension associated with the group of even words is the orientation covering of the foliation.

One can see on Figure 6.1 that the obtained foliation is orientable. The result is actually a torus and its coding yields Sturmian words. Indeed, one way to obtain the orientation covering is to duplicate the alphabet and to work on $(A \cup A') \cup (A \cup A')^{-1}$. Two lifted words are associated with each word: the first one is obtained by replacing the positive

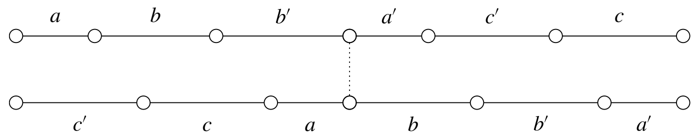

FIGURE 6.2. Interval exchange corresponding to the orientation covering.

letters by elements of A and negative letters by elements of A', and the second one is obtained by replacing the positive letters by letters of $(A')^{-1}$ and the negative ones by elements of A^{-1}. The language of $(A \cup A') \cup (A \cup A')^{-1}$ that is obtained in this way is orientable. As an illustration, the word $c^{-1}ab^{-1}c^{-1}ba^{-1}c$ belongs to the natural coding of T (see Figure 4.2). It admits two lifts that code orbits for the suspension depicted in Figure 6.1, namely $c'ab'c'ba'c'$ and $c^{-1}(a')^{-1}b^{-1}c^{-1}(b')^{-1}a^{-1}c^{-1}$. The word $c'ab'c'ba'c'$ belongs to the natural coding of the interval exchange depicted below. Even letters allow one to stay in the same half of this new interval exchange depicted in Figure 6.2.

The following statement shows a remarkable property of the set of prime words with respect to a subgroup of finite index.

THEOREM 6.9. *Let T be a linear involution relative to the alphabet A without connections and let $S = \mathcal{L}(T)$. For any subgroup G of finite index of the free group F_A, the set of prime words in S with respect to G is a symmetric basis of G.*

Proof of Theorem 6.9. Let T be a linear involution on I relative to the alphabet A without connections. By Lemma 4.5, there exist a measured foliation $(X, \Sigma, \mathcal{F}, \mu)$ and an admissible interval $I \subset X$ such that T is conjugate to the Poincaré map of $\mathcal{F}$ on I. By Lemma 6.1, there is an identification $F_A \to \pi_1(X\backslash\Sigma, x_0)$ for any $x_0 \in I$.

Let G be a subgroup of F_A of index d. By Theorem 6.7, there is a covering $f : \tilde{X} \to X$ of degree d ramified over Σ such that G is identified with $\pi_1(\tilde{X}\backslash f^{-1}(\Sigma))$, i.e. $f_*(\pi_1(\tilde{X}\backslash\Sigma)) = G$.

The pre-image $\tilde{I}$ of the interval I in $\widetilde{X}$ is made of d copies of I. We can also lift the measured foliation to $\widetilde{X}$ and describe the Poincaré map of this measure foliation on $\widetilde{I}$. Indeed, let $\widetilde{I} = \hat{I} \times Q$ where Q is the set of right cosets of G in F_A (recall that $\hat{I} = I \times \{0, 1\}$).

For a point $x \in \hat{I}$, we denote the element of $A \cup A^{-1}$ such that $x \in I_{a(x)}$ by $a(x)$. We define

$$\widetilde{T}(x, Gw) = (Tx, Gwa(x)).$$

Then $\widetilde{T}$ is the Poincaré map of the lift of $(\mathcal{F}, \mu)$ to $\widetilde{X}$ on $\widetilde{I}$.

Now, consider the induced map of $\widetilde{T}$ on the interval $\hat{I} \times \{G\}$ where $\{G\}$ denotes the set reduced to the coset G. For a point $x \in \hat{I}$ we denote by $\rho(x)$ the least $n \geq 1$ such that $\widetilde{T}^n(x, G) \in \hat{I} \times \{G\}$.

We fix a basepoint $\tilde{x}_0$ in $\tilde{X}$ and for a point $x \in \hat{I}$, we denote by $\tilde{\gamma}(x)$ the loop from $\tilde{x}_0$ to itself which passes by $x, T(x), \ldots, T^{\rho(x)-1}(x)$.

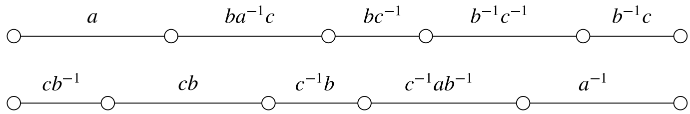

FIGURE 6.3. The transformation induced on the upper level.

The natural coding of a finite orbit $\{x, T(x), \ldots, T^{n-1}(x)\}$ is defined as the word $\Sigma_T^{(n)}(x) = a_0 a_1 \cdots a_{n-1}$ such that $T^i(x) \in I_{a_i}$ for $0 \leq i < n$. Thus, it is the prefix of length n of the infinite natural coding $\Sigma_T(x)$ of T relative to x.

It is easy to verify that the map $\tilde{\gamma}(x) \mapsto \Sigma_T^{(\rho(x))}(x)$ for $x \in \hat{I}$ is a bijection from $\Gamma(\tilde{X}\backslash f^{-1}(\Sigma), \tilde{I}, \tilde{x}_0)$ onto the set of prime words with respect to G which extends to an isomorphism from $\pi_1(\tilde{X}\backslash\Sigma)$ onto G.

By Lemma 6.1, the set $\Gamma(Y\backslash f^{-1}(\Sigma), \tilde{I} \times \{G\})$ is a symmetric basis of G. We thus deduce that the set of prime words with respect to G is a symmetric basis of G. □

COROLLARY 6.10. *Let T be a linear involution without connections. Let w be a word of its natural coding $\mathcal{L}(T)$. The set of right return words to w is a basis of the even group.*

Proof. We assume, without loss of generality, that $I_w \subset I \times \{0\}$. We consider the induced map of T on $I \times \{0\}$. It is an orientable linear involution without connections (that is, an interval exchange with flip(s)), with intervals provided by the prime words of the even group that belong to S_+, with the notation of Proposition 5.5. Furthermore, in the orientable case, the set of complete return words $\mathcal{MR}(w)$ is made of the right return words to w with the left return words to w^{-1}. This conclusion comes from the fact that prime words of the even group that are in S_+ are the right return words to w. □

We illustrate Theorem 6.9 with the group G of even words of the 3-linear involution of Example 4.1. Recall that the covering of degree two of its suspension associated with G is the orientation covering of the foliation (see Example 6.8).

Example 6.11. Let T be as in Example 4.1 and let $S = \mathcal{L}(T)$. Let G be the group of even words in F_A. It is a subgroup of index 2. The set of prime words with respect to G in S is the set $Z \cup Z^{-1}$ with

$$Z = \{a, ba^{-1}c, bc^{-1}, b^{-1}c^{-1}, b^{-1}c\}.$$

Actually, the transformation induced by T on the set $I \times \{0\}$ (the upper part of $\hat{I}$ in Figure 4.1) is the interval exchange transformation represented in Figure 6.3. Its upper intervals are the I_z for $z \in Z$. This relates to the fact that the words of Z correspond to the returns to $I \times \{0\}$ while the words of Z^{-1} correspond to the first returns to $I \times \{1\}$.

Furthermore, one may check directly that the set $Z = \{a, ba^{-1}c, bc^{-1}, b^{-1}c^{-1}, b^{-1}c\}$ is a basis of a subgroup of index 2, in agreement with Theorem 6.9.

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REFERENCES

[**1**] J. Berstel, C. De Felice, D. Perrin, C. Reutenauer and G. Rindone. Bifix codes and Sturmian words. *J. Algebra* **369** (2012), 146–202.

[**2**] V. Berthé, C. De Felice, V. Delecroix, F. Dolce, D. Perrin, J. Leroy, C. Reutenauer and G. Rindone. Specular sets. *Preprint*, 2015, arXiv:1505.00707.

[**3**] V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer and G. Rindone. Bifix codes and interval exchanges. *J. Pure Appl. Algebra* **219** (2015), 2781–2798.

[**4**] V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer and G. Rindone. The finite index basis property. *J. Pure Appl. Algebra* **219** (2015), 2521–2537.

[**5**] V. Berthé, C. De Felice, F. Dolce, J. Leroy, D. Perrin, C. Reutenauer and G. Rindone. Maximal bifix decoding. *Discrete Math.* **338** (2015), 725–742.

[**6**] V. Berthé, C. De Felice, F. Dolce, D. Perrin, J. Leroy, C. Reutenauer and G. Rindone. Acyclic, connected and tree sets. *Monatsh. Math.* **176** (2015), 521–550.

[**7**] V. Berthé and M. Rigo. *Combinatorics, Automata and Number Theory (Encyclopedia of Mathematics and its Applications, 135)*. Cambridge University Press, Cambridge, 2010.

[**8**] C. Boissy and E. Lanneau. Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials. *Ergod. Th. & Dynam. Sys.* **29**(3) (2009), 767–816.

[**9**] A. I. Bufetov and B. Solomyak. On the modulus of continuity for spectral measures in substitution dynamics. *Adv. Math.* **260** (2014), 84–129.

[**10**] T. Coulbois and A. Hilion. Botany of irreducible automorphisms of free groups. *Pacific J. Math.* **256**(2) (2012), 291–307.

[**11**] T. Coulbois, A. Hilion and M. Lustig. $\mathbb{R}$-trees and laminations for free groups. I. Algebraic laminations. *J. Lond. Math. Soc.* (2) **78**(3) (2008), 723–736.

[**12**] C. Danthony and A. Nogueira. Involutions linéaires et feuilletages mesurés. *C. R. Acad. Sci. Paris Sér. I Math.* **307**(8) (1988), 409–412.

[**13**] C. Danthony and A. Nogueira. Measured foliations on nonorientable surfaces. *Ann. Sci. Éc. Norm. Supér* (4) **23**(3) (1990), 469–494.

[**14**] F. Dolce and D. Perrin. Interval exchanges, admissibility and branching Rauzy induction. *Preprint*, 2015.

[**15**] F. Durand. A characterization of substitutive sequences using return words. *Discrete Math.* **179**(1–3) (1998), 89–101.

[**16**] F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergod. Th. & Dynam. Sys.* **20**(4) (2000), 1061–1078.

[**17**] F. Durand. Corrigendum and addendum to: 'Linearly recurrent subshifts have a finite number of non-periodic subshift factors'. [*Ergodic Theory Dynam. Systems* **20**(4) (2000), 1061–1078 MR1779393 (2001m:37022)]. *Ergod. Th. & Dynam. Sys.* **23**(2) (2003), 663–669.

[**18**] A. Fathi, F. Laudenbach and V. Poénaru. *Thurston's Work on Surfaces (Mathematical Notes, 48)*. Princeton University Press, Princeton, NJ, 2012, 254pp.

[**19**] O. Forster. *Lectures on Riemann Surfaces (Graduate Texts in Mathematics, 81)*. Springer, New York, 1991, Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[**20**] D. Gaboriau, G. Levitt and F. Paulin. Pseudogroups of isometries of $\mathbf{R}$: reconstruction of free actions on $\mathbf{R}$-trees. *Ergod. Th. & Dynam. Sys.* **15**(4) (1995), 633–652.

[**21**] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[**22**] M. Keane. Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.

[**23**] L.-M. Lopez and P. Narbel. Lamination languages. *Ergod. Th. & Dynam. Sys.* **33**(6) (2013), 1813–1863.

[**24**] A. Nogueira. Almost all interval exchange transformations with flips are nonergodic. *Ergod. Th. & Dynam. Sys.* **9**(3) (1989), 515–525.

[**25**] A. Nogueira, B. Pires and S. Troubetzkoy. Orbit structure of interval exchange transformations with flip. *Nonlinearity* **26**(2) (2013), 525–537.

[**26**] G. Rauzy. Échanges d'intervalles et transformations induites. *Acta Arith.* **34**(4) (1979), 315–328.

[**27**] A. Skripchenko. Symmetric interval identification systems of order three. *Discrete Contin. Dyn. Syst.* **32**(2) (2012), 643–656.