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On the superimposition of Christoffel words

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ABSTRACT

Initially stated in terms of Beatty sequences, the Fraenkel conjecture can be reformulated as follows: for a k-letter alphabet $\mathcal{A}$, with a fixed $k \geq 3$, there exists a unique balanced infinite word, up to letter permutations and shifts, that has mutually distinct letter frequencies. Motivated by the Fraenkel conjecture, we study in this paper whether two Christoffel words can be superimposed. Following from previous work on this conjecture using Beatty sequences, we give a necessary and sufficient condition for the superimposition of two Christoffel words having the same length, and more generally, of two arbitrary Christoffel words. Moreover, for any two superimposable Christoffel words, we give the number of different possible superimpositions and we prove that there exists a superimposition that works for any two superimposable Christoffel words. Finally, some new properties of Christoffel words are obtained as well as a geometric proof of a classic result concerning the money problem, using Christoffel words.

1. Introduction

Beatty sequences and *Sturmian words* are equivalent objects. The first ones are studied in number theory. The second ones, known since the work of Bernoulli [5], are studied in combinatorics on words and related domains. A *Beatty sequence* is a sequence of the form $S(\alpha, \beta) = \{\lfloor \alpha n + \beta \rfloor : n \in \mathbb{Z}\}$, with $\alpha, \beta \in \mathbb{R}$. It appeared in the literature for the first time in [2], with the name coming over 30 years later in [10,11]. A finite set of Beatty sequences is called an *(eventual) exact cover* if every (sufficiently large) positive integer occurs in exactly one Beatty sequence. It is thus natural to wonder which sets of Beatty sequences are an (eventual) exact cover of the integers. Some particular cases have been studied, for instance in [2,26,3,23,15,20,14,13]. Later, in [12], the Fraenkel conjecture appeared in terms of Beatty sequences, stating that if a finite set of *rational Beatty sequences*, that is Beatty sequences with $\alpha \in \mathbb{Q}$, is an eventual cover of the integers, then the α's satisfy a particular form (see [12] for more details).

In combinatorics on words, the conjecture can be restated as: for a finite k-letter alphabet, with a fixed $k \geq 3$, there exists a unique balanced infinite word, up to letter permutations and shifts, that has mutually distinct letter frequencies. This supposedly unique infinite word is called a *Fraenkel word* and is given by the periodic word Fr_k^{ω}, where Fr_k is defined recursively by $\mathrm{Fr}_k = \mathrm{Fr}_{k-1} k \mathrm{Fr}_{k-1}$ for $k \geq 2$, and $\mathrm{Fr}_1 = 1$.

Particular cases of the Fraenkel conjecture have been extensively studied, for instance by Morikawa, who published a series of papers on the topic (see [25] for a good survey). More precisely, in [19], the author proves the following theorem, which is a necessary and sufficient condition for the disjointness of two Beatty sequences, and that was later reformulated by Simpson [22] as:

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Theorem *([19], See also [22]). Let p_1, p_2, q_1, q_2 be integers and let $p = \gcd(p_1, p_2)$, $q = \gcd(q_1, q_2)$, $u_1 = q_1/q$ and $u_2 = q_2/q$. There exist β_1 and β_2 such that the Beatty sequences $S_1 = \{\lfloor p_1 n/q_1 + \beta_1 \rfloor : n \in Z\}$ and $S_2 = \{\lfloor p_2 n/q_2 + \beta_2 \rfloor : n \in Z\}$ are disjoint if and only if there exist positive integers x and y such that*

$$xu_1 + yu_2 = p - 2u_1u_2(q - 1).$$

This result is a step towards the Fraenkel conjecture. In [22], Simpson works out the proof of Morikawa, gives a new proof and proves some new intermediate results. While translating Simpson's results in terms of Christoffel words, some nice properties of these words appear naturally. In our paper, we first introduce some basic definitions and notation, and we show how the Fraenkel conjecture and the superimposition of Christoffel words are related. Then after having formulated and proved the main results of Simpson in terms of Christoffel words, we go farther and give the number of superimpositions of two Christoffel words, and one possible shift needed to superimpose them. We end this paper by showing how the geometric representation of Christoffel words can be used to prove a problem related to the classical money problem.

2. Preliminaries

We first recall some concepts on words (for more details, see for instance [17]).

An *alphabet* $\mathcal{A}$ is a finite set of symbols called *letters*. A *word over* $\mathcal{A}$ is a sequence of letters from $\mathcal{A}$. The *empty word* ε is the empty sequence. Equipped with the concatenation operation, the set $\mathcal{A}^*$ of *finite words* over $\mathcal{A}$ is a free monoid with neutral element ε and set of generators $\mathcal{A}$, and $\mathcal{A}^+ = \mathcal{A}^* \setminus \varepsilon$. Given a nonempty finite word $u = u[0]u[1]\cdots u[n-1]$, with $u[i] \in \mathcal{A}$, the *length* $|u|$ of u is the integer n. One has $|\varepsilon| = 0$. We denote by $\mathcal{A}^n$ the set of finite words of length n over $\mathcal{A}$ and by $\mathcal{A}^\omega$ the set of *(right-) infinite words* over $\mathcal{A}$. The set $\mathcal{A}^\infty$ is defined as the set of finite and infinite words: $\mathcal{A}^\infty = \mathcal{A}^* \cup \mathcal{A}^\omega$. The set of *bi-infinite words*, denoted by $\mathcal{A}^\mathbb{Z}$, is defined as the set of functions $\mathbb{Z} \to \mathcal{A}$. For the sake of clarity, we denote in bold character a letter denoting an infinite or bi-infinite word, as opposed to a finite word. If $u \in \mathcal{A}^*$, then ${}^\omega u^\omega$ is the bi-infinite word $\mathbf{s} = \cdots u \bullet uu \cdots$. The point $\bullet$ is located between $\mathbf{s}[-1]$ and $\mathbf{s}[0]$ and represents the *origin* of the word $\mathbf{s}$.

As usual, for a finite word u and a positive integer n, the *nth power of u*, denoted u^n, is the word ε if $n = 0$; otherwise $u^n = u^{n-1}u$. The finite word w is *primitive* if it is not the power of a shorter word. If $u \neq \varepsilon$, u^ω (resp. ${}^\omega u$) denotes the right-infinite (resp. left-infinite) word obtained by infinitely repeating u to the right (resp. to the left). A right-infinite word $\mathbf{u}$ is *periodic* (resp. *ultimately periodic*) if it can be written as $\mathbf{u} = w^\omega$ (resp. $\mathbf{u} = vw^\omega$), with $v \in \mathcal{A}^*$ and $w \in \mathcal{A}^+$. The number of occurrences of the letter a in the word u is denoted by $|u|_a$.

Over infinite words, the *shift operator* σ is defined by $\sigma : \mathcal{A}^\mathbb{N} \to \mathcal{A}^\mathbb{N}$ such that $\sigma(\mathbf{s}[n]) = \mathbf{s}[n+1]$. It is also naturally defined over the set of bi-infinite words by $\sigma : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$, with $\sigma(\mathbf{s}[n]) = \mathbf{s}[n+1]$. A shift σ^k, with $k \geq 0$, over a bi-infinite word is equivalent to moving the origin k times to the right. For a letter $\alpha \in \mathcal{A}$ and a finite word $w \in \mathcal{A}^*$, the *conjugacy operator* γ is defined by $\gamma(\alpha w) = w\alpha$. Then $\sigma(w^\omega) = (\gamma(w))^\omega$: γ acts over finite words as the shift σ acts over infinite words.

If, for some words $u, s \in \mathcal{A}^\infty$, $v, p \in \mathcal{A}^*$, $u = pvs$, then v is a *factor* of u, p is a *prefix* of u and s is a *suffix* of u. If $v \neq u$ (resp. $p \neq u$ and $s \neq u$), v is called a *proper factor* (resp. *proper prefix* and *proper suffix*). The *set of factors* of the word u is denoted $F(u)$.

The *reversal* of the finite word $u = u[0]u[1]\cdots u[n-1]$, also called the *mirror image*, is $\tilde{u} = u[n-1]u[n-2]\cdots u[0]$ and if $u = \tilde{u}$, then u is called a *palindrome*. Let $u \in \mathcal{A}^n$. Then $u[i]u[i+1]\cdots u[n-1]u[0]\cdots u[i-1]$ is a *conjugate* of u, for all $0 \leq i \leq n-1$.

In what follows, for $p, q \in \mathbb{N}$, we write $p \perp q$ if $\gcd(p, q) = 1$. Otherwise, we write $p \not\perp q$.

2.1. Christoffel words

In combinatorics on words, instead of using Beatty sequences, we use an equivalent combinatoric object: the Sturmian words. There exists a wide literature about Sturmian words in which we can find several characterizations depending on the context of the study (see for instance [17]). In particular, the Sturmian words are known as the balanced non-periodic infinite words over a 2-letter alphabet. Recall that a finite or infinite word w is *balanced* if for all finite factors $u, v \in F(w)$ having same length and for all letters $a \in \mathcal{A}$, $\big||u|_a - |v|_a\big| \leq 1$.

A finite version of the Sturmian words is the family of Christoffel words. It has been studied for instance in [9,8,4,16]. From the definition of Christoffel words given in [18], in terms of symbolic dynamics, one can easily deduce the following:

Definition 2.1. Let $\mathcal{A} = \{a < x\}$, $\alpha, \beta \in \mathbb{N}$ such that $\alpha \perp \beta$ and let $n = \alpha + \beta$. The *Christoffel word* $u \in \mathcal{A}^*$ with α occurrences of a's and β occurrences of x's is defined by $u = u[0]u[1]\cdots u[n-1]$, where

$$u[i] = \begin{cases} a & \text{if } (i+1)\beta \bmod n > i\beta \bmod n \\ x & \text{if } (i+1)\beta \bmod n \leq i\beta \bmod n \end{cases}$$

for $0 \leq i < n$, where $i\beta \bmod n$ denotes the remainder of the Euclidean division of $i\beta$ by n. We say that u has *slope* β/α.

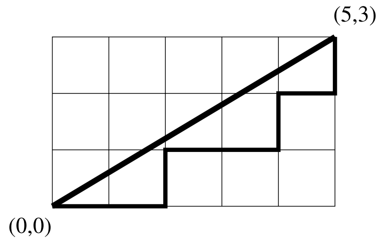
To any Christoffel word, we can associate a Christoffel path, defined as follows.

Definition 2.2. Suppose $p, q \in \mathbb{N}$ and $p \perp q$. The *Christoffel path* of *slope* q/p is the path from $(0, 0)$ to (p, q) in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ that satisfies the following two conditions.

(i) The path lies below the line segment that begins at the origin and ends at (p, q).
(ii) The region in the plane enclosed by the path and the line segment contains no other points of $\mathbb{Z} \times \mathbb{Z}$ besides those of the path.

Notice that the Christoffel word obtained using Definition 2.2 is also called the *lower Christoffel word* and the one obtained by considering the path above the line instead of below is called the *upper Christoffel word*. In this paper, we will only consider the lower ones.

The next figure shows the Christoffel path of slope 3/5.



Notice that Definition 2.1 can be generalized to powers of Christoffel words by removing the condition $\alpha \perp \beta$. We then have:

Definition 2.3. Let $C(n, \alpha)$ be a word of length n over $\{a < x\}^*$ having α occurrences of a's, with $\alpha \leq n$, and let $r = \gcd(n, \alpha)$.

(i) If $r = 1$, then $C(n, \alpha)$ denotes the Christoffel word of slope $\frac{n-\alpha}{\alpha}$.
(ii) If $r > 1$, then $C(n, \alpha) = C(r\frac{n}{r}, r\frac{\alpha}{r}) = \left(C\left(\frac{n}{r}, \frac{\alpha}{r}\right)\right)^r$ denotes the rth power of the Christoffel word $C\left(\frac{n}{r}, \frac{\alpha}{r}\right)$.

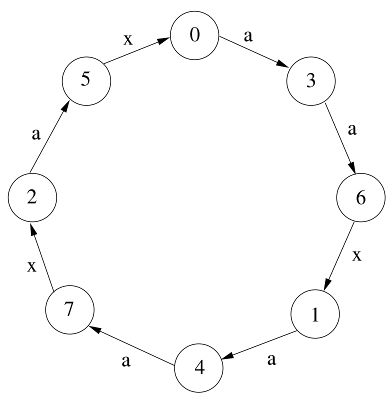
Lemma 2.4. *The reversal of a Christoffel word (resp. power of a Christoffel word) $C(n, \alpha) \in \{a < x\}^*$, denoted by $\widetilde{C}(n, \alpha)$, is also a Christoffel word (resp. power of a Christoffel word) over the same alphabet, but for which the order of the letters is reversed. More precisely, $\widetilde{C}(n, \alpha) = C(n, n - \alpha) \in \{x < a\}^*$.*

Proof. Follows from the fact that the upper Christoffel word is also a Christoffel word, equal to the mirror of the lower Christoffel word (see Proposition 4.2 in [6]). □

Let us now consider the directed graph with the set of vertices $\{0, 1, 2, \ldots, \alpha + \beta - 1\}$ that has an arrow from the vertex i to the vertex j if $i + \beta \equiv j \bmod n$ and labeled by a if $i < j$, and by x if $j < i$.

(i) If $\alpha \perp \beta$, this graph is called the *Cayley graph* of the Christoffel word u over the alphabet $\{a < x\}$ and having slope $\dfrac{\beta}{\alpha}$.
(ii) If $\gcd(\alpha, \beta) = r > 1$, then the graph obtained is isomorphic to the Cayley graph $C\left(\dfrac{\alpha+\beta}{r}, \dfrac{\alpha}{r}\right)$. This graph read r times is the Cayley graph of $C(\alpha + \beta, \alpha)$.

Example 2.5. The Cayley graph associated to the Christoffel word over $\{a < x\}$ having slope 3/5 is



and $u = aaxaaxax$, $|u|_a = 5$, $|u|_x = 3$.

2.2. Link with the Fraenkel conjecture

Before showing how the problem of superimposition of Christoffel words is related to the Fraenkel conjecture, some definitions are required.

First, let us recall that a word $w \in \mathcal{A}^*$ is said to be *circularly balanced* if $w^2 = ww$ is balanced.

Example 2.6. The word $u = 112121$ is balanced, but is not circularly balanced. Indeed, $111, 212 \in F(uu)$, but $\big||111|_1 - |212|_1\big| > 1$.

Example 2.7. One can easily verify that the word 112112 is balanced and consequently, that $v = 112$ is circularly balanced.

Let $w \in \mathcal{A}^*$, with Card $\mathcal{A} \geq 3$. The *projection* $\Pi_a(w)$ of the word $w \in \mathcal{A}^*$ on the alphabet $\{a, x\}$, with $a \in \mathcal{A}$ and $x \notin \mathcal{A}$, is defined by

$$\Pi_a(w)[i] = \begin{cases} a & \text{if } w[i] = a \\ x & \text{otherwise.} \end{cases}$$

Example 2.8. Let $w = 1232343112$. Then $\Pi_1(w) = 1xxxxxx11x$, $\Pi_2(w) = x2x2xxxxx2$, $\Pi_3(w) = xx3x3x3xxx$ and $\Pi_4(w) = xxxxx4xxxx$.

The next result is given without proof, since it is trivial.

Lemma 2.9. *If $w \in \mathcal{A}^*$ is circularly balanced, then for all $a \in \mathcal{A}$, the projection $\Pi_a(w)$ is so.*

Definition 2.10 (*Superimposition of Bi-infinite Words*)**.** Let $\mathbf{u} \in {}^{\omega}\{a, x\}^{\omega}$ and $\mathbf{v} \in {}^{\omega}\{b, x\}^{\omega}$ be two bi-infinite words. Let A be the set of positions of the a's in $\mathbf{u}$ and B be the set of positions of the b's in $\mathbf{v}$. We say that $\mathbf{u}$ and $\mathbf{v}$ are *superimposable* if $A \cap B = \emptyset$.

Example 2.11. Let $\mathbf{u} = {}^{\omega}(aaxaxx)^{\omega}$ and $\mathbf{v} = {}^{\omega}(xxbxxx)^{\omega}$. Then $A = \{0, 1, 3\} + 6\mathbb{Z}$ and $B = \{2\} + 6\mathbb{Z}$. Hence $\mathbf{u}$ and $\mathbf{v}$ are superimposable.

Definition 2.12 (*Superimposition of Finite Words*)**.** Let $u \in \{a, x\}^n$ and $v \in \{b, x\}^m$ be finite words. Let A be the set of positions of the a's in u and let B be the set of positions of the b's in v. Then u and v are *superimposable* if and only if there exists $k \in \mathbb{Z}$ such that ${}^{\omega}u^{\omega}$ and $\sigma^k({}^{\omega}v^{\omega})$ are superimposable, that means if

$$(A + n\mathbb{Z}) \cap (B - k + m\mathbb{Z}) = \emptyset.$$

If $k = 0$, u and v are said to be *perfectly superimposable*.

Remark 2.13. In Definition 2.12, the condition $k \in \mathbb{Z}$ can be replaced by $k \in [0, \min\{m, n\} - 1]$. Indeed, one can easily verify that if there exists a shift k outside this interval that allows the superimposition, then there exists $k' \in [0, \min\{m, n\} - 1]$ such that it is so.

Lemma 2.14. *Let u, v, A and B be such as in Definition 2.12. The words u and v are superimposable and are such that*

$$(A + n\mathbb{Z}) \cap (B - k + m\mathbb{Z}) = \emptyset$$

if and only if u and $\gamma^k(v)$ are perfectly superimposable.

Proof. By definition, u and v are superimposable if and only if there exists $k \in \mathbb{Z}$ such that $(A + n\mathbb{Z}) \cap (B - k + m\mathbb{Z}) = \emptyset$. This condition is satisfied if and only if the set of positions of the a's in ${}^{\omega}u^{\omega}$ and the set of positions of the b's in $\sigma^k({}^{\omega}v^{\omega})$ are disjoint. It is then sufficient to show that the positions of the b's in ${}^{\omega}(\gamma^k(v))^{\omega}$ are the same as in $\sigma^k({}^{\omega}v^{\omega})$. Those two words have period m and, consequently, the positions of the b's are the same if and only if ${}^{\omega}(\gamma^k(v))^{\omega} = \sigma^k({}^{\omega}v^{\omega}) \iff \gamma^k(v) = \sigma^k({}^{\omega}v^{\omega})[0, m-1]$. This last condition is satisfied by the definition of γ. □

Corollary 2.15. *A finite circularly balanced word $w \in \mathcal{A}^m$, with $\mathcal{A} = \{1, 2, \ldots, k\}$, having pairwise distinct letter frequencies can be obtained by the superimposition of k circularly balanced words $w_1 \in \{1, x\}^m$, $w_2 \in \{2, x\}^m, \ldots, w_k \in \{k, x\}^m$ such that $|w_i|_i = |w|_i$ for all $1 \leq i \leq k$.*

Proof. It is sufficient to apply the projection $\Pi_a(w)$ to all letters $a \in \mathcal{A}$ and to conclude using Lemma 2.9. □

Lemma 2.16 ([1])**.** *Any balanced infinite word over a k-letter alphabet, with $k \geq 3$, having pairwise distinct letter frequencies is periodic.*

Corollary 2.15 and Lemma 2.16 give the main motivation of this paper. Lemma 2.16 tells us that in order to prove the Fraenkel conjecture, it is sufficient to prove that for a k-letter alphabet, any circularly balanced finite word having pairwise distinct letter frequencies is a conjugate of Fr_k. Moreover, we deduce from Corollary 2.15 that any finite word satisfying the conditions of the Fraenkel conjecture can be obtained by the superimposition of circularly balanced words, or in other words, by the superimposition of conjugates of powers of Christoffel words, since Christoffel words are primitive balanced words that are minimal with respect to the lexicographic order in their conjugacy class.

In this paper, we are naturally interested in the superimposition of two circularly balanced words. We first only consider two finite primitive words and we give a necessary and sufficient condition for the superimposition of those two words. Moreover, if u is primitive and circularly balanced, then there exists a conjugate of u that is a Christoffel word. Thus, we consider the corresponding Christoffel words w_1 and w_2 and we give a criterion such that w_1 and w_2 are superimposable. To do so, we use results from [22] which are an extension of the works of [19]. We will see that considering a finite circularly balanced word as a conjugate of a power of a Christoffel word allows us to get some nice properties of the Christoffel words.

3. Superimposition of Christoffel words having same length

In this section, we first recall some properties of Christoffel words that will be used in the sequel. Then we study the superimposition of Christoffel words having same length. Notice that most of the results of this section are already known from [19,22], but we include some of their proofs since it will be useful in order to prove the new results presented in the last subsection (number of superimpositions) and in the next sections.

In the sequel, for a positive integer α and a fixed n, we will denote by $\overline{\alpha}$ the integer in $[0 \dots n-1]$ such that $\alpha\overline{\alpha} \equiv -1 \bmod n$.

Lemma 3.1 is a translation of Theorem 3 in [22] in terms of Christoffel words. This result also appears in [4] in an equivalent form using the duality of Christoffel words.

Lemma 3.1 ([22,4]). *Let $C(n,\alpha) \in \{a < x\}^*$ be a Christoffel word. Then the positions of the a's modulo n in $C(n,\alpha)$ are given by the set $\{0, \overline{\alpha}, 2\overline{\alpha}, \dots, (\alpha-1)\overline{\alpha}\}$.*

Lemma 3.1 can be easily generalized to a power of a Christoffel word as:

Corollary 3.2. *Let $C(nq, \alpha q) = (C(n,\alpha))^q$ with $n \perp \alpha$. Then the positions of the a's modulo nq in $C(nq,\alpha q)$ are given by*

$$\bigcup_{i=0}^{q-1} \{0, \overline{\alpha}, 2\overline{\alpha}, \dots, (\alpha-1)\overline{\alpha}\} + in.$$

The following theorem is deduced by Simpson from the Chinese remainder Theorem.

Theorem 3.3 ([22]). *Let $C(n,1)$ and $C(m,1)$ be two Christoffel words. Then $C(n,1)$ and $C(m,1)$ are superimposable if and only if $n \not\perp m$.*

Lemma 3.4. *Let $C(n,\alpha) \in \{a < x\}^*$. For each position i of an a in $\widetilde{C}(n,\alpha)$, there exists $j \in \mathbb{N}$ such that*

$$i\alpha < jn \le (i+1)\alpha.$$

Proof. Recall that by Lemma 2.4, $\widetilde{C}(n,\alpha) = C(n, n-\alpha) \in \{x < a\}^*$. Using the generalization of Definition 2.1 to powers of Christoffel words by replacing respectively a, x, α and β by $x, a, n-\alpha$ and α, we obtain that $\widetilde{C}(n,\alpha)[i] = a$ if and only if $(i+1)\alpha \bmod n \le i\alpha \bmod n$. But $(i+1)\alpha \bmod n \le i\alpha \bmod n$ if and only if there exists a multiple of n between $i\alpha$ and $(i+1)\alpha$ inclusively. This condition is satisfied if and only if there exists j such that $i\alpha < jn \le (i+1)\alpha$. □

Lemma 3.5. *Let $C(n,\alpha) \in \{a < x\}^n$ and $C(n,\beta) \in \{b < x\}^n$ be Christoffel words or power of Christoffel words. If $\alpha | \beta$, then the set of positions of the a's in $C(n,\alpha)$ is a subset of the set of positions of the b's in $C(n,\beta)$.*

Proof. Let us prove this statement for the reversed words. Since $\alpha|\beta$, we write $\beta = q\alpha$, $q \in \mathbb{N}$. Let i be the position of an a in $\widetilde{C}(n,\alpha)$. Then by Lemma 3.4, there exists $j \in \mathbb{N}$ such that $i\alpha < jn \le (i+1)\alpha$. Multiplying both sides of the inequality by q yields

$$i(q\alpha) < (jq)n \le (i+1)(q\alpha) \iff i\beta < (jq)n \le (i+1)\beta,$$

with $jq \in \mathbb{N}$. Hence i is also a position of a b in $\widetilde{C}(n,\beta)$. □

Theorem 3.6 (Th. 2 in [22]). *Let $C(n,\alpha) \in \{a < x\}^n$, $C(m,\beta) \in \{b < x\}^m$ be two Christoffel words and let $p = \gcd(m,n)$. Then $C(n,\alpha)$ and $\gamma^k C(m,\beta)$ are perfectly superimposable if and only if $C(p,\alpha)$ and $\gamma^k C(p,\beta)$ are so.*

Theorem 3.6 is proved in [22] in terms of Beatty sequences. Note that a straightforward proof, in terms of Christoffel words, can be found in [21].

Corollary 3.7. *If the Christoffel words $C(n,\alpha)$ and $C(m,\beta)$ are superimposable, then $m \not\perp n$ and $\alpha+\beta \le p$, with $p = \gcd(m,n)$.*

Proof. By Theorem 3.6, $C(n,\alpha)$ and $C(m,\beta)$ are superimposable if and only if $C(p,\alpha)$ and $C(p,\beta)$ are so. This implies that if $C(n,\alpha)$ and $C(m,\beta)$ are superimposable, then $\alpha+\beta\le p$. Since $\alpha,\beta>0$, we have $1<\alpha+\beta\le p$, and consequently, $m \not| n$. □

In what follows, we will first consider only Christoffel words having the same length, since Theorem 3.6 will then allow us in Section 4 to generalize our results to words of any length.

3.1. Particular case: if $\alpha|\beta$

In this subsection, we study the superimposition of the Christoffel words $C(n,\alpha)$ and $C(n,\beta)$, *having same length*, with $\alpha|\beta$. We give a criterion that the shift must satisfy in order to allow the superimposition (Lemma 3.11), and then we show a necessary and sufficient condition for the superimposition of those Christoffel words (Corollary 3.12). We also exhibit a shift that will always allow the perfect superimposition of two Christoffel words (Corollary 3.13). We end the subsection by showing how a Christoffel word can be viewed as the superimposition of some Christoffel words (Theorem 3.14).

Lemma 3.8. *Let $C(n,\alpha)\in\{a<x\}^n$ be a Christoffel word. Then $C(n,\alpha)=\gamma^{\overline{\alpha}}\widetilde{C}(n,\alpha)$.*

Proof. By Lemma 3.1, the positions of the a's (modulo n) in $C(n,\alpha)$ are given by the set

$$A=\{0,\overline{\alpha},2\overline{\alpha},\dots,(\alpha-1)\overline{\alpha}\}.$$

On the other hand, the positions of the a's (modulo n) in the reverse word $\widetilde{C}(n,\alpha)$ are given by

$$\begin{aligned}\widetilde{A}&=\{n-1,n-1-\overline{\alpha},n-1-2\overline{\alpha},\dots,n-1-(\alpha-1)\overline{\alpha}\}\\&=\{-1,-1-\overline{\alpha},-1-2\overline{\alpha},\dots,-1-(\alpha-1)\overline{\alpha}\}.\end{aligned}$$

We then obtain that the positions of the a's in the conjugate $\gamma^{\overline{\alpha}}\widetilde{C}(n,\alpha)$ are given by

$$\begin{aligned}\gamma^{\overline{\alpha}}\widetilde{A}&=\{-1-\overline{\alpha},-1-2\overline{\alpha},-1-3\overline{\alpha},\dots,-1-\alpha\overline{\alpha}\}\\&=\{(\alpha-1)\overline{\alpha},(\alpha-2)\overline{\alpha},\dots,\overline{\alpha},0\}=A.\quad\square\end{aligned}$$

Definition 3.9. Let $I=[a,b]$ and $I'=[c,d]$ be two intervals of integers. We say that I is located *at the left* of I' if $a<c$.

Lemma 3.10. *The set of differences between the positions of the a's in $C(n,\alpha)\in\{a<x\}^n$ and the positions of the b's in $C(n,\beta)\in\{b<x\}^n$, with $\beta=q\alpha$ and $q\in\mathbb{N}$, forms a set of integers having cardinality $(2\alpha-1)q$.*

Proof. Recall from Lemma 3.1 that the positions of the a's in $C(n,\alpha)$ are $\{0,\overline{\alpha},\dots,(\alpha-1)\overline{\alpha}\}$ and those of the b's in $C(n,\beta)$ are $\{0,\overline{\beta},\dots,(\beta-1)\overline{\beta}\}$. Since $\beta=q\alpha$, multiplying both sides by $\overline{\alpha}\overline{\beta}$ yields $\overline{\alpha}\equiv q\overline{\beta}\bmod n$ and hence $i\overline{\alpha}\equiv iq\overline{\beta}\bmod n$. Consequently, the differences between the positions of the letters form, modulo n, the set

$$E=\{j\overline{\beta}-i\overline{\alpha}\}_{\substack{0\le j<\beta\\0\le i<\alpha}}=\{(j-iq)\overline{\beta}\}_{\substack{0\le j<\beta\\0\le i<\alpha}}.\tag{1}$$

For a fixed i, the possible values of $j-iq$ form the interval $[-iq,\beta-iq[$. Since $q>0$, we deduce that for any i, the interval $[-iq,\beta-iq[$ is at the left of the interval $[-(i-1)q,\beta-(i-1)q[$.

We have $\beta=q\alpha\Rightarrow\beta\ge q\Rightarrow\beta-iq\ge q-iq=-(i-1)q$. Thus, the union of two consecutive intervals is also an interval, and consequently, the union of these α intervals forms the interval $[-(\alpha-1)q,\beta[$, which has cardinality

$$\beta-(-(\alpha-1)q)=\beta+\alpha q-q=\alpha q+\alpha q-q=(2\alpha-1)q.\quad\square\tag{2}$$

Lemma 3.11 (*Th. 4 in [22]*). *Let $C(n,\alpha)\in\{a<x\}^n$ and $C(n,\beta)\in\{b<x\}^n$ be two Christoffel words, with $\beta=q\alpha$ and $q\in\mathbb{N}$ and let $\ell\in[0,n-1]$. The following conditions are equivalent:*

(i) *$C(n,\alpha)$ and $\gamma^{\ell\overline{\beta}}C(n,\beta)$ are perfectly superimposable;*
(ii) *$\ell+n\mathbb{N}\cap[-(\alpha-1)q,\beta[=\emptyset$;*
(iii) *$C(n,\alpha)$ and $\gamma^{\overline{\beta}(1+\ell)}\widetilde{C}(n,\beta)$ are perfectly superimposable.*

Proof. $C(n,\alpha)$ and $\gamma^kC(n,\beta)$ are perfectly superimposable if and only if the shift k is not contained in the set E (see (1)). Otherwise, there is an a in $C(n,\alpha)$ at the same position as a b in $\gamma^kC(n,\beta)$. This last condition is satisfied if and only if there exists $\ell\notin[-(\alpha-1)q,\beta[\bmod n$ such that $k=\ell\overline{\beta}$. Thus there exists $\ell\notin[-(\alpha-1)q,\beta[\bmod n$ if and only if $C(n,\alpha)$ and $\gamma^{\ell\overline{\beta}}C(n,\beta)$ are perfectly superimposable. Hence (i) $\iff$ (ii). Moreover, Lemma 3.8 gives that $C(n,\beta)=\gamma^{\overline{\beta}}\widetilde{C}(n,\beta)$. Replacing $C(n,\beta)$ by this value in $\gamma^{\ell\overline{\beta}}C(n,\beta)$ yields

$$\gamma^{\ell\overline{\beta}}C(n,\beta)=\gamma^{\ell\overline{\beta}}\gamma^{\overline{\beta}}\widetilde{C}(n,\beta)=\gamma^{\overline{\beta}(\ell+1)}\widetilde{C}(n,\beta).$$

Hence (i) $\iff$ (iii). □

Corollary 3.12 (*Cor. 5 in [22]*). *Let $C(n,\alpha)\in\{a<x\}^n$ and $C(n,\beta)\in\{b<x\}^n$ be Christoffel words such that $\beta=q\alpha$, $q\in\mathbb{N}$. Then $C(n,\alpha)$ and $C(n,\beta)$ are superimposable if and only if $(2\alpha-1)q<n$.*

Proof. The words $C(n,\alpha)$ and $C(n,\beta)$ are superimposable if and only if there exists a shift $0 \le k < n$ such that the positions of the α occurrences of a's in $C(n,\alpha)$ form a disjoint set from the set of positions of the β occurrences of b's in $C(n,\beta)$. Such a shift k exists if and only if the set E (from Eq. (1)) has cardinality at most $n-1$. We conclude using the fact that by Eq. (2), Card $(E) = (2\alpha - 1)q$. □

From Lemma 3.11, it is also possible to deduce a shift that always allows the perfect superimposition of two superimposable Christoffel words $C(n,\alpha)$ and $C(n,\beta)$ having same length, with $\alpha|\beta$:

Corollary 3.13. *Let $C(n,\alpha)$ and $C(n,\beta)$ be two superimposable Christoffel words such that $\beta = q\alpha$, $q \in \mathbb{N}$. Then $C(n,\alpha)$ and $\gamma^{(1-r)}\widetilde{C}(n,\beta)$ are perfectly superimposable, with $\alpha r \equiv 1 \bmod n$.*

Proof. By Lemma 3.11 we have $(2\alpha - 1)q < n$ and so

$$2\beta < n + q. \tag{3}$$

Since $\alpha r \equiv 1 \bmod n$ we have $\beta r \equiv q \bmod n$, and (3) implies that no member of the interval $[q+1, 2\beta]$ is congruent to q modulo n. Thus βr does not belong to this interval, and so,

$$\begin{aligned} \beta r - \beta - 1 &\notin [q+1-\beta-1, 2\beta-\beta-1] \\ &= [q - \alpha q, \beta - 1] \\ &= [-(\alpha-1)q, \beta[. \end{aligned}$$

By parts (ii) and (iii) of Lemma 3.11, $C(n,\alpha)$ and $\gamma^{(\beta r-\beta-1)\overline{\beta}}C(n,\beta)$ are superimposable. Using Lemma 3.8 and the fact that γ^n is an identity transformation the second word equals

$$\gamma^{-r+1-\overline{\beta}}C(n,\beta) = \gamma^{1-r}\widetilde{C}(n,\beta)$$

as required. □

Theorem 3.14 (*Th. 6 in [22]*)**.** *Let $C(n,q\alpha) \in \{a < x\}^n$ be a Christoffel word. Then the set of positions of the a's in $C(n,q\alpha)$ is the union of the sets $\{0, \overline{\alpha}, \ldots, (\alpha-1)\overline{\alpha}\} + k\overline{q\alpha}$, for $0 \le k < q$. Moreover, the Christoffel word $C(n,q\alpha)$ is the result of the perfect superimposition of the following q conjugates of $C(n,\alpha)$: $C(n,\alpha), \gamma^{-\overline{q\alpha}}C(n,\alpha), \ldots, \gamma^{-(q-1)\overline{q\alpha}}C(n,\alpha)$.*

Proof. By Lemma 3.1, the set of positions of the a's in $C(n,q\alpha)$ is, modulo n,

$$\{0, \overline{q\alpha}, 2\overline{q\alpha}, \ldots, (q\alpha-1)\overline{q\alpha}\} = \bigcup_{j=0}^{q\alpha-1} j\,\overline{q\alpha} = \bigcup_{k=0}^{q-1}\bigcup_{i=0}^{\alpha-1}(iq+k)\overline{q\alpha}. \tag{4}$$

The last equality is obtained by separating the positions with respect to their remainder modulo q. Since $q\alpha\overline{q\alpha} \equiv -1 \bmod n$, we have $q\overline{q\alpha} \equiv \overline{\alpha} \bmod n$. Thus replacing $q\overline{q\alpha}$ by $\overline{\alpha}$ in Eq. (4) yields

$$\bigcup_{j=0}^{q\alpha-1} j\,\overline{q\alpha} = \bigcup_{k=0}^{q-1}\bigcup_{i=0}^{\alpha-1} i\,\overline{\alpha} + k\overline{q\alpha} = \bigcup_{k=0}^{q-1}\{0, \overline{\alpha}, 2\overline{\alpha}, \ldots, (\alpha-1)\overline{\alpha}\} + k\overline{q\alpha}. \tag{5}$$

We conclude this proof by observing that the q sets $\{0, \overline{\alpha}, \ldots, (\alpha-1)\overline{\alpha}\} + k\overline{q\alpha}$, for $0 \le k \le q-1$, correspond respectively to the positions of the a's in the conjugates of Christoffel words $C(n,\alpha), \gamma^{-\overline{q\alpha}}C(n,\alpha), \ldots, \gamma^{-(q-1)\overline{q\alpha}}C(n,\alpha)$. □

3.2. General case

In this section, we study the general case of the superimposition of two Christoffel words having same length. In order to do so, we consider the Christoffel words $C(n,q\alpha) \in \{a < x\}^*$ and $C(n,q\beta) \in \{b < x\}^*$, with $\alpha \perp \beta$ and $q \in \mathbb{N}$.

Notation 3.15. For $0 \le i < \alpha$, we denote by V_i the interval of integers

$$V_i = [(-q+1)\beta, q\beta - 1] + i\,\overline{\alpha}\beta.$$

Proposition 3.16. *The Christoffel words $C(n,q\alpha)$ and $C(n,q\beta)$ are superimposable if and only if the union*

$$\bigcup_{i=0}^{\alpha-1} V_i \tag{6}$$

is not a complete set of residues modulo n.

Proof. By inverting q and α in Theorem 3.14, we find that $C(n, q\alpha)$ is the perfect superimposition of the α conjugates $C(n, q), \gamma^{-\overline{q\alpha}}C(n, q), \dots, \gamma^{-(\alpha-1)\overline{q\alpha}}C(n, q)$. The set of positions of the a's in $C(n, q\alpha)$ is $\bigcup_{i=0}^{\alpha-1} \mathrm{pos}_a(\gamma^{-i\overline{q\alpha}}C(n, q))$, where $\mathrm{pos}_a(w)$ denotes the positions of the a's in w. Moreover, replacing α, q and β by respectively q, β and $q\beta$ in Lemma 3.11 yields that $C(n, q)$ and $\gamma^{\ell\overline{q\beta}}C(n, q\beta)$ are perfectly superimposable if and only if $\ell \notin [-(q-1)\beta, q\beta[\bmod n$. More generally, $\gamma^{-i\overline{q\alpha}}C(n, q)$ and $\gamma^{\ell\overline{q\beta}}C(n, q\beta)$ are perfectly superimposable if and only if $C(n, q)$ and $\gamma^{\ell\overline{q\beta}+i\overline{q\alpha}}C(n, q\beta)$ are perfectly superimposable. In order to get the form of Lemma 3.11(i), we rewrite $\ell\overline{q\beta} + i\,\overline{q\alpha}$ as

$$\begin{aligned}\ell\overline{q\beta} + i\,\overline{q\alpha} &= \ell\overline{q\beta} - \overline{q\beta}q\beta i\,\overline{q\alpha} \\ &= \overline{q\beta}(\ell + q\beta i\,\overline{q}\,\overline{\alpha}) \\ &= \overline{q\beta}(\ell - i\overline{\alpha}\beta).\end{aligned}$$

We now have the required form of Lemma 3.11(i). Then $\gamma^{\ell\overline{q\beta}+i\overline{q\alpha}}C(n, q\beta)$ and $C(n, q)$ are perfectly superimposable if and only if there exists $\ell - i\overline{\alpha}\beta \notin [-(q-1)\beta, q\beta[\bmod n$. This last condition is equivalent to the existence of a $\ell \notin [-(q-1)\beta, q\beta[+ i\overline{\alpha}\beta = V_i$, but we need that $\ell \notin V_i$ for all $0 \le i < \alpha$. Thus the words $C(n, q\alpha)$ and $C(n, q\beta)$ are superimposable if and only if $\bigcup_{i=0}^{\alpha-1} V_i$ is not a complete set of residues modulo n. □

Corollary 3.17. *There exists $\ell \notin \bigcup_{i=0}^{\alpha-1} V_i \bmod n$ if and only if $C(n, q\alpha)$ and $\gamma^{(\ell+1)\overline{q\beta}}\widetilde{C}(n, q\beta)$ are perfectly superimposable.*

Proof. By Proposition 3.16, the union of the V_i's is not a complete set of residues modulo n if and only if $C(n, q\alpha)$ and $C(n, q\beta)$ are superimposable. Since $C(n, q\alpha)$ is the perfect superimposition of the following α conjugates of $C(n, q)$

$$C(n, q), \gamma^{-\overline{q\alpha}}C(n, q), \dots, \gamma^{-(\alpha-1)\overline{q\alpha}}C(n, q),$$

using the proof of Proposition 3.16 we get that $\gamma^{-i\overline{q\alpha}}C(n, q)$ is perfectly superimposable with $\gamma^{\ell\overline{q\beta}}C(n, q\beta)$ if and only if there exists a $\ell \notin [-(q-1)\beta, q\beta[+ i\overline{\alpha}\beta$ for all $0 \le i < \alpha$. Hence, $C(n, q\alpha)$ and $\gamma^{\ell\overline{q\beta}}C(n, q\beta)$ are perfectly superimposable if and only if there exists $\ell \notin [-(q-1)\beta, q\beta[+ i\overline{\alpha}\beta$ for all $0 \le i < \alpha$. Finally, using Lemma 3.8, $C(n, q\beta) = \gamma^{\overline{q\beta}}\widetilde{C}(n, q\beta)$ and we get that $C(n, q\alpha)$ and $\gamma^{\ell\overline{q\beta}}C(n, q\beta)$ are perfectly superimposable if and only if $C(n, q\alpha)$ and $\gamma^{\ell\overline{q\beta}}\gamma^{\overline{q\beta}}\widetilde{C}(n, q\beta) = \gamma^{(\ell+1)\overline{q\beta}}\widetilde{C}(n, q\beta)$ are so. □

Lemma 3.18. *Let $\alpha, \beta \in \mathbb{N} - \{0\}$, with $\alpha \perp \beta$, and let*

$$x\alpha + y\beta = n - 2\alpha\beta(q-1), \tag{7}$$

with $q, \alpha, \beta \perp n$ and $q \ge 1$. Then:

(i) *Eq. (7) always has a solution $\{x, y\} \in \mathbb{Z}^2$;*
(ii) *it always has a unique solution with $1 \le y \le \alpha$;*
(iii) *if Eq. (7) is satisfied, then $\alpha \perp (\alpha - y)$.*

Proof. Since $\alpha \perp \beta$, Eq. (7) always has a solution $\{x, y\} \in \mathbb{Z}^2$. Let us now suppose that there exist 2 solutions, $\{x, y\}$ and $\{x', y'\}$, such that $1 \le y, y' \le \alpha$. Then $x\alpha + y\beta = x'\alpha + y'\beta$ and consequently, $\alpha(x - x') = \beta(y' - y)$. But $\alpha \perp \beta$ implies that $\alpha | (y' - y)$: this is impossible, since $1 \le y, y' \le \alpha$. Finally, Eq. (7) can be rewritten as

$$\alpha(x + 2\beta(q-1)) = n - y\beta$$

and since $\alpha \perp n$, it follows that $\alpha \perp y$ and hence $\alpha \perp (\alpha - y)$. □

Notation 3.19. In the sequel, let $z = \alpha - y$, where y refers to the solution of Eq. (7). Let $i \in [0, \alpha - 1]$ be one of the possible values of z, as $z = \alpha - y$ and $y \in [1, \alpha]$. Since $\alpha \perp z$ (see Lemma 3.18(iii)), following Simpson [22], there exists a unique $r(i) \in \mathbb{N}$ such that $i \equiv r(i)z \bmod \alpha$. For $0 \le r < \alpha$, let

$$M(r) = r(x + (2q-1)\beta) - \left\lfloor \frac{zr}{\alpha} \right\rfloor \beta. \tag{8}$$

The functions $r(i)$ and $M(r(i))$ will be useful in what follows, in order to obtain a new order for the intervals V_i.

Remark 3.20. Let $a = bq + r$, the Euclidean division of a by b, with $r < b$ and $a, b, q, r \in \mathbb{N}$. We have $r = a \bmod b$ and $q = \left\lfloor \frac{a}{b} \right\rfloor$. Thus,

$$a = bq + r \iff a - bq - r = 0 \iff a - b\left\lfloor \frac{a}{b} \right\rfloor - (a \bmod b) = 0.$$

Lemma 3.21 (*Lemma 7 in [22]*). *For $i \in [0, \alpha - 1]$, $M(r(i)) \equiv -i\overline{\alpha}\beta \bmod n$.*

Proof. For a fixed i, let us consider $\alpha(M(r(i)) + i\overline{\alpha}\beta)$. In what follows, we will write r instead of $r(i)$, in order to simplify the notation. Using Eq. (7) and the definition of $M(r)$, we get:

$$\alpha\left(M(r) + i\overline{\alpha}\beta\right) \equiv \beta\left(zr - \alpha\left\lfloor\frac{zr}{\alpha}\right\rfloor - i\right) \bmod n.$$

Replacing a and b in the previous remark by respectively zr and α and using the fact that $i \equiv zr \bmod \alpha$ yields that the term in parenthesis has value 0 and consequently, that $\alpha(M(r) + i\overline{\alpha}\beta) \equiv 0 \bmod n$. Since $\alpha \perp n$, $M(r) + i\overline{\alpha}\beta \equiv 0 \bmod n$ and we conclude. □

Lemmas 3.22–3.26 are not original results, since they appeared without emphasis in the proof of Theorem 8 in [22]. However, they are the key for the proofs of the results in the next section.

Lemma 3.22. *Let $n \in \mathbb{N}$ be a fixed integer and let $I_0, I_1, \ldots, I_{r-1}$ be r finite intervals in $\mathbb{Z}$ having same length and satisfying:*

(i) $\max(I_0) - \min(I_{r-1}) \geq n - 1 \geq 1$;
(ii) *for* $0 \leq j < r - 1$, *if I_{j+1} is located at the left of I_j, then $I_{j+1} \cup I_j$ is an interval.*

Then $\bigcup_{j=0}^{r-1} I_j$ is a complete set of residues modulo n.

Proof. Let us suppose that the interval I_{r-1} is not located at the left of the interval I_0. Since $\max(I_0) - \min(I_{r-1}) \geq n - 1$, it implies that $I_0 \cup I_{r-1}$ is an interval and that $I_0 \cap I_{r-1} = [\min(I_{r-1}), \max(I_0)]$. It follows that Card $(I_0 \cap I_{r-1}) \geq n$ and that $\bigcup_{j=0}^{r-1} I_j$ is a complete set of residues modulo n. Let us now suppose that the interval I_{r-1} is located at the left of the interval I_0. By (ii), there exist consecutive intervals that are located one to the left of the others. Condition (ii) also ensures that all the integers between I_{r-1} and I_0 are in the union of the j intervals. Since $\max(I_0) - \min(I_{r-1}) \geq n - 1$, the number of integers between the beginning of the interval I_{r-1} and the end of the interval I_0 is at least n. In both cases, $\bigcup_{j=0}^{r-1} I_j$ is a complete set of residues modulo n. □

Lemma 3.23. *Let $I_0, I_1, \ldots, I_{r-1}$ be finite intervals in $\mathbb{Z}$ and let I be the shortest interval that contains them. Let us suppose that*

(i) $I \setminus \bigcup_{j=0}^{r-1} I_j$ *is non-empty;*
(ii) *if* $x \in \bigcup_{j=0}^{r-1} I_j$ *and* $y \in I \setminus \bigcup_{j=0}^{r-1} I_j$, *then* $|y - x| < n$.

Then $\bigcup_{j=0}^{r-1} I_j$ does not contain all the integers modulo n.

Proof. Let $y \in I \setminus \cup_{j=0}^{r-1} I_j$. By (ii) there can be no $x \in I$ such that $x \equiv y \bmod n$ and the result follows. □

Lemma 3.24. *Let $z = \alpha - y$ and $0 \leq r < \alpha$ be such as in Notation 3.19. Then* $\left\lfloor\frac{z(r+1)}{\alpha}\right\rfloor - \left\lfloor\frac{zr}{\alpha}\right\rfloor \in \{0, 1\}$.

Proof. Let $zr = i\alpha + t$, with $i \in \mathbb{N}$ and $0 \leq t < \alpha$. Then

$$\left\lfloor\frac{z(r+1)}{\alpha}\right\rfloor - \left\lfloor\frac{zr}{\alpha}\right\rfloor = \left\lfloor\frac{i\alpha + t + z}{\alpha}\right\rfloor - \left\lfloor\frac{i\alpha + t}{\alpha}\right\rfloor = i + \left\lfloor\frac{t + z}{\alpha}\right\rfloor - i - \left\lfloor\frac{t}{\alpha}\right\rfloor = \left\lfloor\frac{t + z}{\alpha}\right\rfloor - \left\lfloor\frac{t}{\alpha}\right\rfloor.$$

Since $1 \leq y \leq \alpha$ and $0 \leq t < \alpha$, we have $0 \leq t + z < 2\alpha$ and consequently,

$$\left\lfloor\frac{t + z}{\alpha}\right\rfloor - \left\lfloor\frac{t}{\alpha}\right\rfloor = \left\lfloor\frac{t + z}{\alpha}\right\rfloor - 0 \leq 1. \quad \square$$

Lemma 3.25. *Let $M(r)$ be defined as in Eq. (8). Then*

(i) $M(0) = 0$;
(ii) $M(\alpha - 1) = n - x - 2\beta(q - 1) - i$, *with $i = 0$ if $y \neq \alpha$ and $i = \beta$ otherwise.*

Proof. (i) $M(0) = 0(x + (2q - 1)\beta) - \left\lfloor\frac{z \cdot 0}{\alpha}\right\rfloor\beta = 0$.
(ii) If $y \neq \alpha$, then

$$M(\alpha - 1) = (\alpha - 1)(x + (2q - 1)\beta) - \left\lfloor\frac{z(\alpha - 1)}{\alpha}\right\rfloor\beta \tag{9}$$

$$= \alpha x + (2q - 1)\alpha\beta - x - (2q - 1)\beta - z\beta + \beta \tag{10}$$

$$= \alpha x + y\beta - \alpha\beta + (2q - 1)\alpha\beta - x + \beta - (2q - 1)\beta$$

$$= n - x - 2\beta(q - 1). \tag{11}$$

Eq. (10) is deduced from Eq. (9) using the fact that

$$\left\lfloor\frac{z(\alpha - 1)}{\alpha}\right\rfloor\beta = z\beta + \left\lfloor\frac{-z}{\alpha}\right\rfloor\beta = z\beta - \beta,$$

since $0 < z = \alpha - y < \alpha$, as $1 \le y < \alpha$, while Eq. (11) is obtained using Eq. (7).

If $y = \alpha$, then $z = 0$ and

$$\left\lfloor \frac{z(\alpha-1)}{\alpha} \right\rfloor \beta = 0,$$

implying

$$M(\alpha-1) = n - x - 2\beta(q-1) - \beta. \quad \square \tag{12}$$

Lemma 3.26. *Let $M(r)$ be defined as in Eq. (8). Then*

(i) $M(r+1) - M(r) = x + (2q-1)\beta - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right)$;

(ii) *if $x \le 0$, then $M(r+1) - M(r) \le \beta(2q-1)$.*

Proof. We have:

$$\begin{aligned} M(r+1) - M(r) &= \left((r+1)(x+(2q-1)\beta) - \left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor \beta\right) - \left(r(x+(2q-1)\beta) - \left\lfloor \frac{zr}{\alpha} \right\rfloor \beta\right) \\ &= x + (2q-1)\beta - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right), \end{aligned}$$

which is $\le \beta(2q-1)$ if $x \le 0$, using Lemma 3.24. $\square$

The following theorem is a particular case of Theorem 8 in [22] which first appeared in [19].

Theorem 3.27 ([19,22]). *$C(n, q\alpha)$ and $C(n, q\beta)$ are superimposable if and only if there exists $\{x, y\} \in \{\mathbb{N} - \{0\}\}^2$ such that*

$$x\alpha + y\beta = n - 2\alpha\beta(q-1). \tag{13}$$

Proof. For the Christoffel words $C(n, q\alpha)$ and $C(n, q\beta)$, by Lemma 3.18 there exists a unique $\{x, y\}$ satisfying Eq. (13), with $1 \le y \le \alpha$. We want to show that $C(n, q\alpha)$ and $C(n, q\beta)$ are superimposable if and only if $x > 0$.

($\Longrightarrow$) Let us suppose that $x \le 0$ and let us consider the union of the intervals given in Eq. (6). Using Lemma 3.21, we have, modulo n,

$$\bigcup_{i=0}^{\alpha-1} \{[(-q+1)\beta, q\beta-1] + i\bar{\alpha}\beta\} = \bigcup_{r=0}^{\alpha-1} \{[(-q+1)\beta, q\beta-1] - M(r)\}. \tag{14}$$

Let $I_r = [(-q+1)\beta, q\beta-1] - M(r)$, for $0 \le r < \alpha$. Then using Lemma 3.25, we get $\max(I_0) = q\beta - 1 - M(0) = q\beta - 1$ and $\min(I_{\alpha-1}) = (-q+1)\beta - M(\alpha-1)$ and then

$$\begin{aligned} \max(I_0) - \min(I_{\alpha-1}) &= q\beta - 1 - ((-q+1)\beta - M(\alpha-1)) \\ &= q\beta - 1 + q\beta - \beta + n - x - 2\beta(q-1) - i \\ &= \beta + n - x - 1 - i, \end{aligned}$$

where $i = 0$ if $y \ne \alpha$ and $i = \beta$ otherwise (see Lemma 3.25). Since $x \le 0$, $-x$ is non-negative. Hence $\max(I_0) - \min(I_{\alpha-1}) \ge n - 1$.

If $I_r \cup I_{r+1}$ is not an interval then $M(r+1) - M(r) > |I_r| + 1$. Thus, in order to show that $I_r \cup I_{r+1}$ is an interval, it is sufficient to show that $M(r+1) - M(r) \le |I_r| + 1$. By Lemma 3.26 (ii), we have $M(r+1) - M(r) \le \beta(2q-1)$. Moreover, all the intervals have length

$$|I_r| = q\beta - 1 - (-q+1)\beta = 2q\beta - \beta - 1 = \beta(2q-1) - 1.$$

Hence $M(r+1) - M(r) \le |I_r| + 1$. Then, for all $0 \le r < \alpha - 1$, $I_r \cup I_{r+1}$ is an interval. Recall that Proposition 3.16 tells us that $C(n, q\alpha)$ and $C(n, q\beta)$ are superimposable if and only if the union (14) is not a complete set of residues modulo n. Applying Lemma 3.22, we conclude that for $x \le 0$ the words are not superimposable.

($\Longleftarrow$) Let us now suppose that $x > 0$ and let us show that it implies that the words are superimposable. By Proposition 3.16, it is sufficient to show that if $x > 0$, then $\bigcup_{r=0}^{\alpha-1} \{[-(q-1)\beta, q\beta-1] - M(r)\}$ does not contain all the integers modulo n.

Let us recall that $I_r = [-(q-1)\beta, q\beta-1] - M(r)$, for $0 \le r < \alpha$. Since $x > 0$ and $q \ge 1$, and using Lemmas 3.24 and 3.26, we have

$$\begin{aligned} M(r+1) - M(r) &= x + (2q-1)\beta - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right) \\ &\ge x + (2q-1)\beta - \beta \\ &= x + 2\beta(q-1) \ge x > 0. \end{aligned}$$

Thus, the intervals I_r are located one to the left of the others, for $0 \le r < \alpha$. They all have the same cardinality, that is: $\text{Card}\,(I_r) = |I_r| + 1 = \beta(2q-1) - 1 + 1 = \beta(2q-1)$.

Let us suppose that $I = \bigcup_{r=0}^{\alpha-1} I_r$ is not an interval. Then condition (i) of Lemma 3.23 is satisfied. For condition (ii), it is sufficient to take $y = \min(I_0) - 1$ (since $\max(I \setminus \cup I_j) \le \min(I_0) - 1$) and $x = \min(I)$ and to check that $y - x < n$. We have $x = -(q-1)\beta - M(\alpha - 1)$ and $y = -(q-1)\beta - 1$.

Consequently:

$$y - x = (-(q-1)\beta - 1) - (-(q-1)\beta - (n - x - 2\beta(q-1) - i)) = n - x - 1 - 2\beta(q-1) - i,$$

with $i \in \{0, \beta\}$. Since $x > 0$, this value is $< n$. By Lemma 3.23, we conclude that the union of these intervals does not contain all the integers modulo n. □

3.3. Number of superimpositions of Christoffel words

In this section, we prove the exact number of superimpositions of two Christoffel words having same length and we give a shift that always allows a perfect superimposition for two superimposable Christoffel words.

Definition 3.28. Let $C(n, \alpha)$ and $C(n, \beta)$ be two superimposable Christoffel words. The *number of superimpositions* of these two words is defined by

$$\text{Card}\ \left(\{k \in [0, n-1] \mid C(n, \alpha) \text{ and } \gamma^k C(n, \beta) \text{ are perfectly superimposable}\}\right).$$

Some results are first required.

Corollary 3.29 (*Of Lemma 3.23 and of its Proof*)**.** *If the two conditions of Lemma 3.23 are satisfied, then*

(i) *the elements of $I \setminus \bigcup_{j=0}^{r-1} I_j$ are all distinct modulo n;*

(ii) *if $\text{Card}\,(I) \ge n$, then modulo n, the elements of $I \setminus \bigcup_{j=0}^{r-1} I_j$ are exactly the ones that are not in $\bigcup_{j=0}^{r-1} I_j$;*

(iii) *if $\text{Card}\,(I) < n$, then modulo n, the elements of $\mathbb{Z} \setminus \bigcup_{j=0}^{r-1} I_j$ are exactly the ones that are in $I \setminus \cup_{j=0}^{r-1} I_j$ and the following $n - \text{Card}\,(I)$ elements:*

$$\{\min(I) - (n - \text{Card}\,(I)), \ldots, \min(I) - 2, \min(I) - 1\}.$$

Proof. (i) Let $x, y \in I \setminus \bigcup_{j=0}^{r-1} I_j$ and without loss of generality, let us suppose that $y > x$. Then $y \le \max(I \setminus \bigcup_{j=0}^{r-1} I_j)$ and $x > \min(I)$, since $\min(I)$ is contained in $\bigcup_{j=0}^{r-1} I_j$. Consequently $y - x < \max(I \setminus \bigcup_{j=0}^{r-1} I_j) - \min(I)$ which is, by Lemma 3.23(ii), $< n$. Hence $y - x < n$.

(ii) Since I is an interval and $\text{Card}\,(I) \ge n$, I contains all the elements mod n. By Lemma 3.23(ii), there is no element in $\bigcup_{j=0}^{r-1} I_j$ that is equal, modulo n, to an element in $I \setminus \bigcup_{j=0}^{r-1} I_j$.

(iii) One can easily observe that the $n - \text{Card}\,(I)$ elements are not equal, modulo n, to any element of I. □

Lemma 3.30. *Let $C(n, j) \in \{a < b\}^*$ be a Christoffel word. Then*

$$C(n,j)[i] = \begin{cases} a & \text{if } \left\lfloor \dfrac{n-j}{n}(i+1) \right\rfloor - \left\lfloor \dfrac{n-j}{n} i \right\rfloor = 0 \\[2ex] b & \text{if } \left\lfloor \dfrac{n-j}{n}(i+1) \right\rfloor - \left\lfloor \dfrac{n-j}{n} i \right\rfloor = 1. \end{cases}$$

Proof. Follows from the definition of Christoffel words. Taking the difference between the integer parts corresponds to checking if there is a multiple of n or not between both values. If the difference is 0, then no multiple of n occurs. □

Proposition 3.31. *Let $C(n, q\alpha)$ and $C(n, q\beta)$ be two superimposable Christoffel words. The number of superimpositions of $C(n, q\alpha)$ and $C(n, q\beta)$ is*

(i) *xy, if $x \le \beta$;*

(ii) *$x\alpha + y\beta - \alpha\beta$, if $x > \beta$;*

where $\{x, y\}$ is the unique solution of Eq. (13), with $1 \le y \le \alpha$.

Proof. Let us recall from Theorem 3.27 that if two Christoffel words are superimposable and if the solution of Eq. (13) is $\{x, y\}$ with $1 \le y \le \alpha$, then $x > 0$. Let us denote by I the shortest interval that contains the union of the intervals given in Eq. (14). Then using Lemma 3.25, we get

$$\begin{aligned}\text{Card}\ (I) &= \max(I) - \min(I) + 1\\ &= \max(I_0) - \min(I_{\alpha-1}) + 1\\ &= (q\beta - 1) - ((-q+1)\beta - M(\alpha-1)) + 1\\ &= q\beta - 1 + q\beta - \beta + (n - x - 2\beta(q-1) - i) + 1\\ &= n - x + \beta - i,\end{aligned}$$

with $i = 0$ if $y \ne \alpha$, and $i = \beta$ otherwise.

(i) Let us suppose that $x \le \beta$ and $y \ne \alpha$. Then $\text{Card}\ (I) = n - x + \beta \ge n$. By Corollary 3.29(ii), the complementary set modulo n of $\bigcup_{j=0}^{\alpha-1} I_j$ has the same cardinality as the number of elements contained between I_0 and I_1, I_1 and I_2, etc. The number of elements contained between I_r and I_{r+1} is $M(r+1) - M(r) - (2q-1)\beta$, that is the distance between the beginning of both intervals minus the cardinality of one interval. Using Lemma 3.26, we have

$$M(r+1) - M(r) - (2q-1)\beta = x - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right). \tag{15}$$

There is a gap between two intervals if the value of (15) is > 0. This value corresponds to the number of integers contained in the gap. Since $x \le \beta$, it will be the case for all r such that $\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor = 0$. Using Lemma 3.30, with $j = y$, $i = r$ and $n = \alpha$, we find that it is the case for exactly y values of r. Thus, there are xy possible superimpositions.

(ii) Let us suppose that $x > \beta$. One can easily observe that $x > \beta \Longrightarrow y \ne \alpha$. Then $\text{Card}\ (I) = n - x + \beta < n$. We still have that $\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor = 0$ for y values of r. Moreover, since $0 \le r < \alpha$, there are $(\alpha - 1)$ gaps each containing

$$x - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right)$$

integers. Thus, by Lemma 3.30,

$$\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor = 1$$

for $\alpha - 1 - y = z - 1$ values. Hence $x - \beta\left(\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor\right) = x - \beta$ for $(z-1)$ values of r. Using Corollary 3.29(iii), we know that there are $n - \text{Card}\ (I)$ others possible values outside the interval I. The number of superimpositions is then given by

$$\begin{aligned} xy + (x-\beta)(z-1) + n - \text{Card}\ (I) &= xy + (x-\beta)(\alpha - y - 1) + n - (n - x + \beta)\\ &= x\alpha + y\beta - \alpha\beta.\end{aligned}$$

(iii) Let us suppose that $x \le \beta$ and $y = \alpha$. Then $\text{Card}\ (I) = n - x < n$. This case is similar to case (ii), except that here, since $y = \alpha$,

$$\left\lfloor \frac{z(r+1)}{\alpha} \right\rfloor - \left\lfloor \frac{zr}{\alpha} \right\rfloor = 0$$

between every interval and hence $z = \alpha - y = 0$. Since there are $(y-1) = (\alpha-1)$ gaps between I_0 and $I_{\alpha-1}$, using Corollary 3.29(iii), we find that the number of possible superimpositions is given by

$$x(y-1) + n - \text{Card}\ (I) = x(y-1) + n - (n-x) = xy. \quad \square$$

Remark 3.32. Since Lemma 3.18, we have supposed that $\{x, y\}$ is the solution of Eq. (13) such that $y \le \alpha$. In the proof of Proposition 3.31, we still use this assumption. It is possible to rewrite all these results considering the solution for which $x \le \beta$. We would have obtained a similar result as in Proposition 3.31, with the conditions $y \le \alpha$ and $y > \alpha$.

Theorem 3.33 is a generalization of Corollary 3.13 for any values of q, α, β, such that $\alpha \perp \beta$: for two superimposable Christoffel words having same length, we give a shift that always allows a perfect superimposition.

Theorem 3.33. *Let $C(n, q\alpha)$ and $C(n, q\beta)$ be two superimposable Christoffel words, with $\alpha \perp \beta$. Then $C(n, q\alpha)$ and $\gamma^{1-r}\widetilde{C}(n, q\beta)$ are perfectly superimposable, where $qr \equiv 1 \bmod n$.*

Proof. By Corollary 3.17, $C(n, q\alpha)$ and $\gamma^{(\ell+1)\overline{q\beta}}\widetilde{C}(n, q\beta)$ are superimposable if and only if $\exists \ell \notin \bigcup_{i=0}^{\alpha-1} V_i \bmod n$. It is then sufficient to show that there exists $\ell \notin \bigcup_{i=0}^{\alpha-1} V_i \bmod n$ such that $(\ell + 1)\overline{q\beta} \equiv 1 - r$. Isolating ℓ, we get that this last condition is equivalent to

$$\ell \equiv -q\beta + rq\beta - 1 \equiv \beta - 1 - q\beta.$$

Let us show that $\beta - 1 - q\beta \notin \bigcup_{i=0}^{\alpha-1} V_i \bmod n$.

If $\alpha = 1$, by Eq. (14), the union of the V_i's is the interval $[-q\beta + \beta, q\beta - 1]$. Then $\beta - q\beta - 1$ is the element preceding the interval and since the words are superimposable, the interval has a length $< n$, and consequently $\beta - q\beta - 1 \bmod n$ is not contained in the interval.

If $\alpha > 1$, let us consider the intervals I_0 and I_1. There exist elements between both intervals, since

$$M(1) - M(0) - (2q-1)\beta = x + (2q-1)\beta - \left\lfloor \frac{z}{\alpha} \right\rfloor \beta - 0 - (2q-1)\beta = x > 0,$$

as $z = \alpha - y < \alpha$. Moreover,

$$\begin{aligned}]\max(I_1), \min(I_0)[&=]q\beta - 1 - (x + (2q-1)\beta), (-q+1)\beta[\qquad (16)\\ &=]q\beta - 1 - x - 2q\beta + \beta, -q\beta + \beta[\\ &=]\beta - q\beta - 1 - x, -q\beta + \beta[. \qquad (17)\end{aligned}$$

Thus, $\beta - q\beta - 1$ is located between I_0 and I_1. In order to conclude, it is sufficient to show that this element does not appear in an other interval. It is true if $(\beta - q\beta - 1) - \min(I_{\alpha-1}) < n$. Let us verify:

$$\begin{aligned} (\beta - q\beta - 1) - \min(I_{\alpha-1}) &= \beta - q\beta - 1 - ((-q+1)\beta - (n - x - 2\beta(q-1) - i)) \\ &= n - 2\beta(q-1) - x - 1 - i < n \end{aligned}$$

where $i \in \{0, \beta\}$. □

4. Generalization to words having different lengths

In this section, we use Theorem 3.6 in order to generalize the results of Sections 3.2 and 3.3 for arbitrary Christoffel words, not necessarily having same length.

Theorem 4.1 ([19,22])**.** *Let $C(n, q\alpha)$ and $C(m, q\beta)$ be Christoffel words, with $\alpha \perp \beta$. Then $C(n, q\alpha)$ and $C(m, q\beta)$ are superimposable if and only if there exists $\{x, y\} \in \{\mathbb{N} - \{0\}\}^2$ such that*

$$x\alpha + y\beta = p - 2\alpha\beta(q-1), \qquad (18)$$

with $p = \gcd(m, n)$.

Proof. By Theorem 3.6, $C(n, q\alpha)$ and $C(m, q\beta)$ are superimposable if and only if $C(p, q\alpha)$ and $C(p, q\beta)$ are so. We conclude using Theorem 3.27, since it ensures that $C(p, q\alpha)$ and $C(p, q\beta)$ are superimposable if and only if there exists $\{x, y\} \in \{\mathbb{N} - \{0\}\}^2$ satisfying Eq. (18). □

Lemma 4.2. *If $C(p, \alpha)$ and $\gamma^{-k}C(p, \beta)$ are perfectly superimposable, then $C(n, \alpha)$ and $\gamma^{-k+ip}C(m, \beta)$ are so, with $m > n$, $p = \gcd(n, m)$ and $0 \le i < \frac{m}{p}$.*

Proof. Theorem 3.6 shows that $C(p, \alpha)$ and $\gamma^{-k}C(p, \beta)$ are perfectly superimposable if and only if $C(n, \alpha)$ and $\gamma^{-k}C(m, \beta)$ are so. Moreover, one can easily observe that $C(p, \alpha)$ and $\gamma^{-k}C(p, \beta)$ are perfectly superimposable if and only if $C(p, \alpha)$ and $\gamma^{-k+ip}C(p, \beta)$ are so. These $-k + ip$ correspond to different shifts, for $0 \le i < \frac{m}{p}$, for words of length at most m. □

Proposition 4.3. *Let $C(n, q\alpha)$ and $C(m, q\beta)$ be two superimposable Christoffel words, with $\alpha \perp \beta$, $p = \gcd(m, n)$ and $m > n$. The number of superimpositions is*

(i) $xy\dfrac{m}{p}$, *if $x \le \beta$;*

(ii) $(x\alpha + y\beta - \alpha\beta)\dfrac{m}{p}$, *if $x > \beta$;*

with $\{x, y\} \in \{\mathbb{N} - \{0\}\}^2$ the solution of $x\alpha + y\beta = p - \alpha\beta(q-1)$ such that $y \le \alpha$.

Proof. Follows from Proposition 3.31 and from Lemma 4.2. □

Theorem 4.4. *Let $C(n, q\alpha)$ and $C(m, q\beta)$ be two superimposable Christoffel words, with $\alpha \perp \beta$ and $p = \gcd(m, n)$. Then $C(n, q\alpha)$ and $\gamma^{-(r-1)+ip}\widetilde{C}(m, q\beta)$ are perfectly superimposable, with $qr \equiv 1 \bmod p$ and $0 \le i < \dfrac{m}{p}$.*

Proof. Follows from Theorem 3.33 and from Lemma 4.2. □

5. Other results

In this last section, we first give a new necessary and sufficient condition for the perfect superimposition of two Christoffel words $C(n,\alpha)$ and $C(n,\beta)$, with $\alpha \perp \beta$. Then we give a result concerning the word obtained by the superimposition of two Christoffel words having the same length. We end this section with a new proof of a problem related to the money problem, using the geometric interpretation of Christoffel words.

Theorem 5.1. *Let $u = C(n,\alpha) \in \{a < z\}^n$ and $v = C(n,\beta) \in \{b < z\}^n$ be Christoffel words. There exists $\{x,y\} \in \{\mathbb{N} - \{0\}\}^2$ such that $\alpha x + \beta y = n$ if and only if u and $\widetilde{v}$ are perfectly superimposable.*

Proof. ($\Longrightarrow$) Let us suppose that there exists $\{x,y\} \in \{\mathbb{N} - \{0\}\}^2$ such that $\alpha x + \beta y = n$. Let us consider the Christoffel words $u' = C(n, x\alpha)$ and $v' = C(n, y\beta)$. Since $\alpha x + \beta y = n$, these words are complementary, that means that u' and $\widetilde{v}'$ are perfectly superimposable. Using Lemma 3.5, we conclude that u and $\widetilde{v}$ are so.

($\Longleftarrow$) Let us suppose that u and $\widetilde{v}$ are perfectly superimposable. Let $d = \gcd(\alpha,\beta)$. By Lemma 3.5, $C(n,d) \in \{a < z\}^n$ and $\widetilde{C}(n,d) \in \{z < b\}^n$ are also superimposable. Let us now show that u and $\widetilde{v}$ are superimposable only if $d|n$. If $d \nmid n$, then $C(n,d)$ can be written as the product of az^i and az^{i+1}, it begins by az^i and ends by az^{i+1}. Moreover, $\widetilde{C}(n,d)$ ends by bz^ib. There is a conflict between a letter a and a letter b, since

$$C(n,d) = paz^iz \quad \text{and} \quad \widetilde{C}(n,d) = p'bz^ib.$$

Thus, if the words are perfectly superimposable, $d|n$.

Moreover, since $d = \gcd(\alpha,\beta)$, $d|\alpha$ and $d|\beta$. Thus, $\gcd(n,\alpha) = d$ and $\gcd(n,\beta) = d$. Since $C(n,\alpha)$ and $C(n,\beta)$ are Christoffel words, $\alpha \perp n$ and $\beta \perp n$. Hence $d = 1$. Applying Theorem 4.1 with $m = n, q = 1$, we get $p = 1$ and consequently, $C(n,\alpha)$ and $C(n,\beta)$ are superimposable if and only if there exists $\{x,y\} \in \{\mathbb{N} - \{0\}\}^2$ such that $x\alpha + y\beta = n$. □

Definition 5.2. Let $w \in \{a,b\}^*$ and let $A = \{i_1, i_2, \ldots, i_k\}$ be the set of positions of the a's in w. Then the word $w' \in \{a,b\}^*$ obtained by the *decimation* $D_{p/q,a}$ of w over the letter a, with $p \le q$, is the word w for which we have deleted the letters $w[i_j]$, for all $j \in \{\ell q+1, \ell q+2, \ldots, \ell q+p\}_{0\le \ell \le \lfloor |A|/q \rfloor}$ if $p/q < 0$ and for all $j \in \{|A|-\ell q, |A|-\ell q-1, \ldots, |A|-\ell q-p+1\}_{0\le \ell\le \lfloor |A|/q\rfloor}$ otherwise. In other words, w' is the word w for which p occurrences over q of the letter a are removed from left to right if $p/q < 0$, and from right to left otherwise.

Example 5.3. Let us consider $w = aabaababababa$. The decimation $D_{1/3,a}(w)$ yields $w' = abababab$. Then performing $D_{-1/2,b}$ over w' gives $w'' = aabaab$.

Theorem 5.4. *Let $u = C(n,\alpha) \in \{a < z\}^n$ and $C(n,\beta) \in \{b < z\}^n$ be two superimposable Christoffel words with $\alpha \perp \beta$. Let v be the conjugate of $C(n,\beta)$ that is perfectly superimposable to u. Let w be defined as*

$$w[i] = \begin{cases} a & \text{if } u[i] = a \\ b & \text{if } v[i] = b \\ z & \text{otherwise.} \end{cases}$$

Let w' be the word obtained from w, after having removed the letter z. Then w' is the Christoffel word of slope β/α.

Proof. Let $\{x,y\} \in \{\mathbb{N} - \{0\}\}^2$ be such that $\alpha x + \beta y = n$. By Theorem 3.27, we know that such x, y exist. Let us consider the Christoffel word $t \in \{a < b\}^n$ with αx occurrences of the letter a and βy occurrences of the letter b. Let us perform the decimation $D_{(x-1)/x,a}(t)$: it removes $(\alpha x - \alpha)$ letters a's. The decimation $D_{-(y-1)/y,b}$ over the word obtained removes $(\beta y - \beta)$ letters b's. Since the decimation operation preserves Christoffel words [7] and since a couple of number of occurrences of letters determines a unique Christoffel word, the word w' obtained is the Christoffel word of length $\alpha + \beta$ with α occurrences of the letter a and β occurrences of the letter b. □

Example 5.5. Let $u = C(13,4) = azzzazzazzzazzz$ and $C(13,3) = bzzzbzzzbzzzz$. These words are superimposable. Indeed, it is sufficient to take the conjugate $v = \widetilde{C}(13,3) = zzzzbzzzbzzzb$. We then find $w = azzabzazbazzb$ and $w' = aababab$. Note that the equation $4x + 3y = 13$ has the solution $x = 1$ and $y = 3$. Thus, we consider the Christoffel word $t = C(13,4) = abbabbabbabbb$. The decimation $D_{0/1,a}(t)$ does not erase any a. Then we perform $D_{-2/3,b}$ over $D_{0/1,a}(t)$: starting from the left we erase 2 occurrences over 3 of b's. We get $aababab = C(7,4)$.

5.1. Money problem

In Theorem 5.1, we showed that two Christoffel words u and $\tilde{v}$ of length n are perfectly superimposable if and only if there exist integers α, β such that $\alpha x + \beta y = n$. In what follows, $\alpha x + \beta y$ occurs again: we prove, using the geometric interpretation of Christoffel words, classical results of Sylvester concerning the money problem, also known as the *Frobenius problem*.

Let us first recall the money problem.

Definition 5.6 (*[27]*)**.** Let $0 < a_1 < \ldots < a_n$ be n integers, with $n \geq 2$, that represent n different values of money pieces and such that $\gcd(a_1, a_2, \ldots, a_n) = 1$. The possible amounts of money that can be obtained using these n pieces are given by

$$\sum_{i=1}^{n} a_i x_i,$$

where $x_i \in \mathbb{N}$ denotes the number of the piece a_i used. The *money changing problem* consists of determine the greatest integer $N = g(a_1, a_2, \ldots, a_n)$ that cannot be obtained using the pieces of money $a_1, a_2, \ldots, a_n$. This integer is called the *Frobenius number*.

If $a_1 = 1$, all amounts can be obtained. It is not the case in general: only a few amounts can be obtained. For instance, with pieces of 2, 5 and 10, it is impossible to obtain 1 and 3, while all the other quantities can be obtained. Hence $g(2, 5, 10) = 3$.

Proposition 5.7 (*[24]*)**.** *The greatest integer that cannot be obtained with the pieces a and b is*

$$g(a, b) = (a - 1)(b - 1) - 1. \tag{19}$$

Proposition 5.8 appears in [27], but the origin is unknown.

Proposition 5.8. *The number of integers that cannot be obtained with the pieces a and b is given by*

$$\frac{(a-1)(b-1)}{2}. \tag{20}$$

Corollary 5.9. *The number of elements of the submonoid of N generated by a and b and smaller than* $(a-1)(b-1)$ *is* $\frac{(a-1)(b-1)}{2}$.

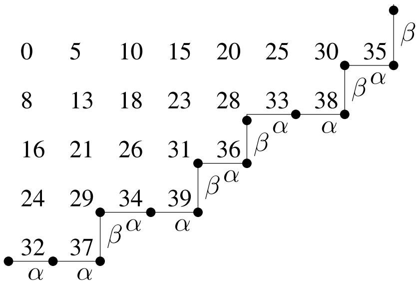
Proof. We know by Proposition 5.7 that all the integers greater or equal to $(a-1)(b-1)$ are representable with a and b. Thus the unrepresentable $\frac{(a-1)(b-1)}{2}$ integers given in Proposition 5.8 are necessarily smaller than $(a-1)(b-1)$. Since half of the $(a-1)(b-1)$ elements smaller than $(a-1)(b-1)$ (including the 0) are not representable with a and b, there is exactly the same quantity that is representable. □

In what follows, we will show that it is possible to prove Corollary 5.9 using the geometric representation of Christoffel words and their Cayley graphs.

Theorem 5.10. *Let* $a, b \in \mathbb{N}$. *Let us consider the quadrant defined by* $x \geq 0$ *and* $y \leq 0$, *having at the coordinate* $(x, -y)$ *the value* $xb + ya$. *While considering only the integer coordinates* $(x, -y)$ *such that* $xb + ya < ab$, *the boundary obtained can be coded by a Christoffel word having exactly a occurrences of the letter* α *and b occurrences of the letter* β.

Here is first an example of Theorem 5.10.

Example 5.11. For $a = 8$, $b = 5$, we have $ab = 40$. We then get:



Associating the letter α to a move to the right and the letter β to a move to the top, and if we start at the lower leftmost corner, the lower boundary is coded by the word $\alpha\alpha\beta\alpha\alpha\beta\alpha\beta\alpha\alpha\beta\alpha\beta$: it is the Christoffel word with 8 occurrences of α and 5 occurrences of β.

Proof of Theorem 5.10. Let us consider the Cayley graph of the Christoffel word with a occurrences of the letter α and b occurrences of the letter β, with $\alpha < \beta$. We get the Cayley graph linearly represented by

$$0 \to b \to 2b \bmod (a+b) \to \ldots \to ib \bmod (a+b) \to \ldots \to (a+b-1)b \bmod (a+b) \to 0$$

In this Cayley graph, if there exists $k \in \mathbb{N}$ such that

$$ib < k(a+b) \leq (i+1)b,$$

then

$$(i+1)b \bmod (a+b) = (ib \bmod (a+b)) - a. \tag{21}$$

Otherwise, we have

$$(i+1)b \bmod (a+b) = (ib \bmod (a+b)) + b. \tag{22}$$

Let us consider the preceding Cayley graph to which we add the value $ab-a-b$. Since the values in the initial Cayley graph were lower than or equal to $a+b$, the values in the new Cayley graph are now lower than or equal to $a+b+ab-a-b=ab$. This corresponds exactly to taking the lower and rightmost path such that the value of the coordinate $(x, -y)$ is lower than or equal to ab. Indeed, we do $+b$ (see Eq. (22): right move) if we exceed the value ab, otherwise we do $-a$ (see Eq. (21): up move). □

In the preceding example, the Cayley graph is

$$0 \to 5 \to 10 \to 2 \to 7 \to 12 \to 4 \to 9 \to 1 \to 6 \to 11 \to 3 \to 8 \to 0.$$

The new Cayley graph obtained by adding $ab-a-b=27$ is

$$27 \to 32 \to 37 \to 29 \to 34 \to 39 \to 31 \to 36 \to 28 \to 33 \to 38 \to 30 \to 35 \to 27.$$

and corresponds to the boundary described in Example 5.11.

Here is a new proof of Corollary 5.9 that uses the result of Theorem 5.10.

Proof of Corollary 5.9. Excluding the integers that are on the boundary in Theorem 5.10 and using the Cayley graph seen previously, we obtain that there are exactly $xa+yb$ integers that are lower than $(a-1)(b-1)$. The total number of elements in the rectangle is ab and since we have to remove the boundary which contains $a+b-1$ elements, and divide by 2, we obtain: $\frac{ab-(a+b-1)}{2} = \frac{(a-1)(b-1)}{2}$. □

6. Concluding remarks

In this paper, we have expressed in term of words combinatorics, a necessary and sufficient condition for the superimposition of two Christoffel words, by translating the results of [19,22] in terms of Christoffel words. For two superimposable Christoffel words, we did more than in [19,22] by giving a possible shift that always allows the perfect superimposition of two superimposable Christoffel words and the number of possible shifts. Those results are interesting since they give new properties of the well-known Christoffel words. Finally, in order to prove the Fraenkel conjecture, it would be interesting to generalize the condition for the superimposition of Christoffel words for more than two Christoffel words. Since the condition for the superimposition does not tell us which shift is necessary to superimpose the two Christoffel words, and moreover, since for n Christoffel words, $\binom{n}{2}$ Diophantine equations are necessary in order to make sure that the n Christoffel words are eventually superimposable, this last problem appears to be a challenging one.

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