On the Three-Distance Theorem

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The three-distance theorem states that if one chooses a real number α and a natural number n, then the ordered sequence whose elements are the fractional parts of the points $0, \alpha, 2\alpha, \ldots, (n-1)\alpha$, together with 1, partitions the unit interval [0, 1] into successive intervals that have at most three different lengths. Moreover (icing on the cake), if there are three lengths, then the longest one is the sum of the other two.

This beautiful theorem was conjectured by Steinhaus, and first proved in 1958 by Sós [15], Surányi [19], and Świerczkowski [16], and then by Slater [17] and Halton [8]. See also [1, 9, 18, 22] for more on the subject.

One richness of the three-distance theorem is that it is simultaneously a combinatorial, arithmetic, and dynamical statement. This is reflected in the variety both of its proofs and of its generalizations. Arithmetically, this theorem has to do with the approximation of irrational numbers by rational numbers, based on the approximation properties of the intermediate partial convergents in the continued fraction expansion of α. Dynamically, this theorem can be interpreted in terms of lattices; see, for instance, [13], which relies on homogeneous dynamics and on the properties of the space of two-dimensional Euclidean lattices.

The aim of the present paper is to combine the combinatorial and dynamical viewpoints by focusing on the finite words that encode the successive lengths. Note that such a combination has culminated in recent higher-dimensional generalizations such as those developed in [2, 6, 7].

The Three-Distance Theorem

Let α be a real number. We denote by $\{\alpha\}$ its fractional part; in other words, it is $\alpha \mod 1$. And let n be a natural number. According to the three-distance theorem, the successive intervals of [0, 1] obtained by placing the points of the reordered sequence $\{i\alpha\}$, $i = 0, \ldots, n-1$, together with 1 have at most three distinct lengths, and if there are three lengths, then the longest one is the sum of the other two. We use the term "distances" for these lengths in order to avoid any confusion with the lengths of the many intervals that will occur in this article.

Let us consider the example illustrated in Figure 1, where the points are represented on a circle instead of the segment [0, 1], since they are naturally points of $\mathbb{R}/\mathbb{Z}$. Take $\alpha = 5/22$ and $n = 7$. Multiplying everything by 22, we replace [0, 1] by [0, 22], and we consider the multiples $5i$ of 5 modulo 22, for $i = 0, \ldots, 6$, which are 0, 5, 10, 15, 20, 3, 8, and we order them, which gives 0, 3, 5, 8, 10, 15, 20. The successive intervals of [0, 22] thus obtained are [0, 3], [3, 5], [5, 8], [8, 10], [10, 15], [15, 20], [20, 22], of successive lengths 3, 2, 3, 2, 5, 5, 2.

Our task now will be to show that the sequence of lengths of the successive intervals described above forms a

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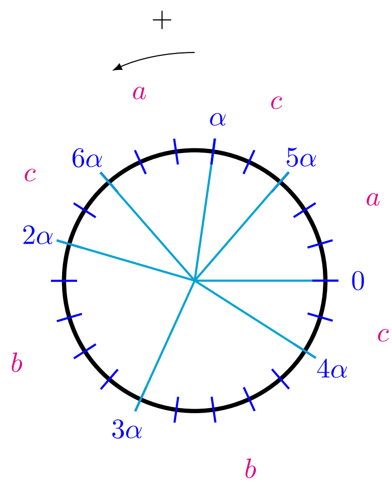

Figure 1. An illustration of the three-distance theorem, with $\alpha = 5/22$ and $n = 7$, where the three lengths are encoded by the letters a, b, c. Its associated permutation in its cycle form is $\sigma = (0, 5, 1, 6, 2, 3, 4)$.

word that belongs to a very special class of words, namely, word encodings of three-interval exchanges, equivalently, perfectly clustering words.

For the example considered above, the word is *acacbbc*, where the length of the interval starting at 0 is coded by the letter a, the longest one by b, and the remaining one by c (see Figure 1).

We consider α to be either irrational or rational, but in the latter case, to avoid repetitions in the sequence $\{i\alpha\}$, one makes the hypothesis that n is smaller than the smallest positive denominator of α. Thus, one obtains n successive intervals when considering the points $0, \{\alpha\}, \{2\alpha\}, \ldots, \{(n-1)\alpha\}$, together with 1. We will now work on the unit interval rather than on the unit circle, such as depicted in Figure 1. This will be in line with the permutations that we will consider.

Denote by $0 = x_0 < x_1 < \cdots < x_{n-1}$ the numbers $i\alpha$ mod 1 reordered (with $0 \leq i \leq n-1$); let $x_n = 1$. We also define, for $0 \leq i \leq n-1$, k_i as the unique integer in

$$[\![n]\!] = \{0, 1, \ldots, n-1\}$$

such that $k_i\alpha \equiv x_i$ modulo 1. Its uniqueness comes either from the fact that α is irrational or from the assumption that n is smaller than the smallest positive denominator of α. Our aim will be to express the successor map that sends a point $k_i\alpha$ to its right neighbor; that is, it maps x_i to x_{i+1}. We will describe this map explicitly as a permutation σ of $[\![n]\!]$ defined by $k_i \mapsto k_{i+1}$. For the example of Figure 1, the permutation is $\sigma = (0, 5, 1, 6, 2, 3, 4)$ in its cycle form.

Circular Symmetric Discrete Interval Exchanges and Their Word Encoding

Let $n \geq 1$. Let $(c_1, \ldots, c_\ell)$ be a composition of n, that is, an ℓ-tuple of natural integers whose sum is n (for convenience, we allow zeros in the composition). We decompose the interval $[\![n]\!] = \{0, 1, 2, \ldots, n-1\}$ into intervals in two ways: the intervals $I_1, \ldots, I_\ell$ (respectively $J_1, \ldots, J_\ell$) are defined by the condition that they are consecutive and that $|I_j| = c_j$ (respectively $|J_h| = c_{\ell+1-h}$). Denote by S_n the group of permutations of $[\![n]\!]$. We define the permutation $\sigma \in S_n$ by the condition that it sends increasingly each interval I_h onto the interval $J_{\ell+1-h}$. We call such a permutation a symmetric discrete interval exchange,[1] and it will be said to be associated with the composition $(c_1, \ldots, c_\ell)$.

A symmetric discrete interval exchange may be equivalently defined using local translations. The permutation σ indeed acts on $[\![n]\!]$ as a discrete version of an interval exchange (see, e.g., [4]) as described now.

We do it for three intervals. So let $\ell = 3$ and let (c_1, c_2, c_3) be the composition, with $c_1 + c_2 + c_3 = n$. Then the permutation σ is defined by

$$\sigma(i) = \begin{cases} i + c_2 + c_3 & \text{if } i \in I_1 = \{0, \ldots, c_1 - 1\}, \\ i + c_3 - c_1 & \text{if } i \in I_2 = \{c_1, \ldots, c_1 + c_2 - 1\}, \\ i - c_1 - c_2 & \text{if } i \in I_3 = \{c_1 + c_2, \ldots, n-1\}. \end{cases} \quad (1)$$

As an example, consider the 3-tuple $(2, 2, 3)$. The intervals I_1, I_2, I_3 are $\{0, 1\}, \{2, 3\}, \{4, 5, 6\}$, and the intervals J_1, J_2, J_3 are $\{0, 1, 2\}, \{3, 4\}, \{5, 6\}$; σ sends increasingly $\{0, 1\}$ onto $\{5, 6\}$, $\{2, 3\}$ onto $\{3, 4\}$, and $\{4, 5, 6\}$ onto $\{0, 1, 2\}$, whence

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 0 & 1 & 2 \end{pmatrix},$$

and its cycle form is $(0, 5, 1, 6, 2, 3, 4)$. Observe that it corresponds to the permutation of our example from Figure 1.

We say that the permutation σ is circular if it has only one cycle. The previous example is circular. Note that it was proved by Pak and Redlich that the probability that a given symmetric discrete exchange of three intervals in S_n is circular tends to $1/\zeta(2) = 6/\pi^2$ as n tends to ∞ [11].

Let us return to the general case. For a circular symmetric exchange σ of ℓ intervals, we define its word encoding as follows: with the previous notation, let $\{a_1 < \cdots < a_\ell\}$ be a totally ordered alphabet; let σ in cycle form be $(0, k_1, \ldots, k_{n-1})$; replace in the word $0k_1 \ldots k_{n-1}$ each digit k by a_j if $k \in I_j$.

In the example, with the alphabet $\{a < b < c\}$, we obtain *acacbbc* as the encoding word. We see that this word is the same word as the one from the example of Figure 1. This will be explained in the next section.

[1] The word "symmetric" refers to the fact that the intervals are exchanged according to the central symmetry of the set $\{1, 2, \ldots, \ell\}$, that is, the mapping $h \mapsto \ell + 1 - h$.

A Theorem

Consider n successive closed intervals of $[0, 1]$ whose union is $[0, 1]$, like those considered in the three-distance theorem. Call the lengths of these intervals "distances." Suppose that there are two or three distances, with the condition that in the latter case, the leftmost interval is not the longest one. We define the distance encoding as the word on the alphabet $\{a, b, c\}$ obtained by replacing the distances from left to right as follows: if there are three distances, code the leftmost distance (the length of the interval starting at 0) by a, the longest distance by b, and the other one by c. If there are two distances, code the leftmost by a and the other by c.

Theorem 1. *Take a nonzero real number α and a natural integer $n \geq 1$ as explained before, that is, if α is rational, then n is smaller than the smallest positive denominator of α. Suppose that there are three distances for the successive intervals of $[0, 1]$ obtained by placing the points of the reordered sequence $\{i\alpha\}$, $i = 0, \ldots, n-1$, together with 1. Then the leftmost interval is not the longest one. The distance encoding of these intervals is the word encoding of some circular symmetric exchange of three intervals on the alphabet $\{a < b < c\}$.*

Theorem 1 is essentially [23, Theorem 2], but we provide here a purely combinatorial proof. We prove this theorem in the next section, first when α is rational, and then we deduce it for irrational α by a compactness argument, or, in combinatorial terms, by applying the pigeonhole principle. As a byproduct, we obtain a new proof of the three-distance theorem; this will be seen in the course of the proof.

We first give an idea of the proof by continuing our running example, with $\alpha = 5/22$, $n = 7$, where we multiply everything by $N = 22$ as explained above: we replace $[0, 1]$ by $[0, 22]$, and we consider the multiples $5i$ of 5 modulo 22, for $i = 0, \ldots, 6$. See also Figure 1 for an illustration. We found that the seven successive intervals of $[0, 22]$ are $[0, 3]$, $[3, 5]$, $[5, 8]$, $[8, 10]$, $[10, 15]$, $[15, 20]$, $[20, 22]$, with sequence of lengths 3, 2, 3, 2, 5, 5, 2. Indeed, the first seven multiples of 5 modulo 22 are 0, 5, 10, 15, 20, 3, 8, and reordered, they are 0, 3, 5, 8, 10, 15, 20.

We now list all numbers from 0 to 21 (that is, all numbers modulo 22), and put in boldface the previous list:

(**0**, 1, 2, **3**, 4, **5**, 6, 7, **8**, 9, **10**, 11, 12, 13, 14, **15**, 16, 17, 18, 19, **20**, 21).

This emphasizes the distances, which are the gap lengths determined by the boldfaced elements. Note that the inverse modulo 22 of 5 is 9. Multiplying modulo 22 everything by 9, we obtain

(**0**, 9, 18, **5**, 14, **1**, 10, 19, **6**, 15, **2**, 11, 20, 7, 16, **3**, 12, 21, 8, 17, **4**, 13).

Multiplying by 9 allows one to consider elements of $[\![N]\!]$ and $[\![n]\!]$ as acting as multiples of α, i.e., with the previous notation, to shift from $x_i = k_i\alpha$ to k_i. We now consider this sequence as a circular permutation ω; this map is nothing else than addition of 9 modulo 22. Now the boldfaced elements refer to the sequence (k_i) such that the elements $k_i\alpha$ occur successively in $[0, 1]$.

Let us now "induce" ω with respect to $[\![7]\!]$, by considering σ the cyclic restriction of ω to $[\![7]\!]$, that is, the permutation obtained by removing in the cycle ω all elements that are not in $[\![7]\!]$. Then $\sigma = (0, 5, 1, 6, 2, 3, 4)$.

It turns out that σ is a discrete exchange of three intervals; indeed, it is the permutation σ of our running example. Moreover, the three distances (that is, the lengths of the runs) are the numbers $s(i)$, $i = 0, \ldots, 6$, where $s(i)$ is the smallest exponent s such that $\omega^s(i) \in \{0, 1, 2, 3, 4, 5, 6\}$. In other words, the three distances are the return times of addition of 9 modulo 22.

We see that the word encoding of σ is *acacbbc*, which corresponds precisely to the sequence of distances 3, 2, 3, 2, 5, 5, 2. Note that any irrational number close enough to 5/22 gives the same permutation and the same word. We have chosen to give here an example with α rational, since it allows integer computations.

Proof of the Theorem

The proof is divided into three parts, which we label A, B, C. In part A, we formally define cyclic restrictions and their basic properties. In part B, we treat the case of α rational, and α irrational is treated in part C. A lemma used in the proof is given at the end of the section.

A. 1. We begin by defining formally the cyclic restriction of a permutation. Given a permutation $\omega \in S_N$ and n, $0 \leq n \leq N-1$, we define the cyclic restriction σ of ω to $[\![n]\!]$ as follows: take a cyclic representation of ω and remove from it all elements $\geq n$ (this construction does not depend on the chosen cyclic representation). Clearly, this does not increase the number of cycles of ω, and if ω is circular, so is σ.

2. For $i = 0, \ldots, n-1$, let $s(i)$ denote the smallest positive exponent such that $\omega^{s(i)}(i) \in [\![n]\!]$. This exponent exists by the definition of σ. Then one has $\sigma(i) = \omega^{s(i)}(i)$ for $i = 0, \ldots, n-1$.

B. We first prove the theorem when α is rational. Write $\alpha = r/N$, in reduced form. Since the computations are modulo 1, we may assume that $\alpha \in [0, 1)$; then $r < N$, and we also have $n < N$ by hypothesis.

1. As was illustrated in the running example, we multiply everything by N: instead of considering the sequence $i\alpha$ modulo 1, $i = 0, \ldots, n-1$, in $[0, 1]$, we consider the sequence ir mod N in $[0, N]$, together with the number N, and we obtain n intervals of $[0, N]$. Reordering this sequence, we denote the new sequence by x_j, $j = 0, \ldots, n-1$; we define $k_j \in [\![n]\!]$ as the unique number in $[\![N]\!]$ such that $x_j \equiv k_j r \mod N$. We also let $x_n = N$. Then the distances in the theorem are the numbers $x_{i+1} - x_i$, $i = 0, \ldots, n-1$.

2. Let $q \in [\![N]\!]$ be the inverse of r modulo N. We define a permutation $\omega \in S_N$: it is addition by q modulo N. Thus one has $\omega^i(0) \equiv qi$ modulo N, for all i.

We see that ω is a circular symmetric exchange of two intervals in S_N. Indeed, ω sends the interval $\{0, \ldots, N-q-1\}$ (respectively $\{N-q, \ldots, N-1\}$) increasingly onto the interval $\{q, \ldots, N-1\}$ (respectively

$\{0, \ldots, q-1\}$). Thus ω is associated with the composition $(N-q, q)$.

3. We let σ denote the cyclic restriction of ω to $[\![n]\!]$ and show that $x_i = r\sigma^i(0)$ for all $i = 0, \ldots, n-1$. Note indeed that the sequence x_i, $i = 0, \ldots, n-1$, is increasing, and that the underlying set of this sequence is $\{ir, i = 0, \ldots, n-1\}$, where the computations are modulo N. Moreover, these two properties characterize the sequence.

Hence it is enough to show that the sequence $r\sigma^i(0)$, $i = 0, \ldots, n-1$, has these two properties.

By the definition of σ, its cycle form $(0, \sigma(0), \ldots, \sigma^{n-1}(0))$ is obtained from the cycle form $(0, \omega(0), \ldots, \omega^{N-1}(0))$ of ω by removing from the latter the elements $\omega^i(0)$ satisfying $n \le \omega^i(0)$, $i = 0, \ldots, N-1$. Hence the sequence $\sigma^i(0)$, $i = 0, \ldots, n-1$, is a subsequence of the sequence $\omega^i(0)$, $i = 0, \ldots, N-1$. It follows that the sequence $r\sigma^i(0) \bmod N$, $i = 0, \ldots, n-1$, is a subsequence of $r\omega^i(0) \bmod N$, $i = 0, \ldots, N-1$. But $r\omega^i(0) \equiv riq \equiv i \bmod N$, since q is the inverse of r modulo N; hence the sequence $r\sigma^i(0) \bmod N$, $i = 0, \ldots, n-1$, is a subsequence of $0, 1, \ldots, N-1$, and it is therefore an increasing sequence, which proves the first property.

The set of numbers $\sigma^i(0)$, $i = 0, \ldots, n-1$, coincides with $[\![n]\!]$; hence

$$\{r\sigma^i(0) \mod N,\ i = 0, \ldots, n-1\} \\ = \{jr \mod N,\ j = 0, \ldots, n-1\},$$

which proves the second property.

4. For the argument below, it is useful to have the following definition: let $a_0, a_1, \ldots, a_{N-1}$ be a sequence with subsequence $b_0, b_1, \ldots, b_{n-1}$, with $a_0 = b_0$. Call each subsequence $b_j = a_i, a_{i+1}, \ldots, a_{k-1}$, where $a_k = b_{j+1}$, or $k = N$ if $j = n-1$, a gap. We call the length of this latter subsequence the gap length.

We know that the distances in the theorem are the numbers $x_{i+1} - x_i$, $i = 0, \ldots, n-1$. In other words, they are the gap lengths determined by the subsequence $0 = x_0, x_1, \ldots, x_{n-1}$ of the sequence $0, 1, \ldots, N-1$. Since $x_i = r\sigma^i(0)$, we see, by multiplying by q modulo N, that these gap lengths are identical for the subsequence $0, \sigma(0), \ldots, \sigma^{n-1}(0)$ of the sequence $0, \omega(0), \ldots, \omega^{N-1}(0)$. Hence the sequence of these gap lengths is $s(\sigma^i(0))$, $i = 0, \ldots, n-1$ (with s as defined in part A.2), which is therefore the sequence of distances.

5. We now claim that σ is a circular symmetric discrete exchange of two or three intervals. The claim will follow from Lemma 2, given after the proof.

Let (c_1, c_2, c_3) be the composition corresponding to σ, with $c_1, c_3 > 0$, and $c_2 \ge 0$ (the case $c_2 = 0$ corresponding to the case in which σ is an exchange of two intervals). As done in our running example, we let I_1, I_2, I_3 be the successive intervals of $[\![n]\!]$ of lengths c_1, c_2, c_3. We then have (1).

Since $\sigma(i) = \omega^{s(i)}(i)$ by part A.2, we have $\sigma(i) \equiv i + s(i)q \mod N$. Since q is relatively prime to N, we obtain that if $i \in I_1$, then $s(i)$ is the unique solution in $[\![N]\!]$ of $s(i)q \equiv c_2 + c_3 \mod N$. Precisely, $s(i) \equiv r(c_2 + c_3) \mod N$ by (1), which, together with the condition $s(i) \in [\![N]\!]$, completely determines $s(i)$ for $i \in I_1$. Similarly, if $i \in I_2$, then $s(i) \equiv r(c_3 - c_1) \mod N$, and if $i \in I_3$, then $s(i) \equiv -r(c_1 + c_2)$.

Hence s takes at most three values, and there are at most three interval lengths. This proves the three-distance theorem for α rational.

6. We now use the hypothesis of Theorem 1, namely, that there are exactly three distances. Thus s takes three values, and we denote them by $s(i) = s_1, s_2, s_3$ for $i \in I_1, I_2, I_3$ respectively. Then modulo N, one has $s_1 + s_3 - s_2 \equiv r(c_2 + c_3 - c_1 - c_2 - c_3 + c_1) = 0$. Since $s_1 + s_2 + s_3 < N$, we must have $s_2 = s_1 + s_3$. In particular, the maximum of s_1, s_2, s_3 is s_2.

The sequence of distances is $s(\sigma^i(0))$, $i = 0, \ldots, n-1$. Since $0 \in I_1$, its first element $s(0) = s_1$ is not the maximum of the distances, which proves the first assertion. Moreover, since $\sigma^i(0) \in I_j \Leftrightarrow s(\sigma^i(0)) = s_j$, the word encoding of σ is the word corresponding to the sequence of distances.

C. Suppose now that α is an irrational real number. Let (α_k), for $k = 0, 1, 2, \ldots$, be a sequence of rational numbers whose limit is α (in what follows, the limit will be always as $k \to \infty$). We may assume that the smallest denominators of the α_k are all larger then n. For $i = 0, \ldots, n-1$, the limit of $i\alpha_k \mod 1$ is $i\alpha \mod 1$ (this is clear for $i = 0$, and for $i \neq 0$, $i\alpha$ is not an integer, since α is irrational, so that there is some neighborhood of $i\alpha$ where the function $\lfloor x \rfloor$ is constant and where the function $x \mod 1 = x - \lfloor x \rfloor$ is therefore continuous). It follows that for k large enough, the relative order of the numbers $i\alpha_k \mod 1$, $i = 0, \ldots, n-1$, is the same as the relative order of the numbers $i\alpha$, $i = 0, \ldots, n-1$. Thus the sequence of distances in $[0, 1]$ determined by the numbers $i\alpha_k \mod 1$, $i = 0, \ldots, n-1$, tends to the sequence of distances in $[0, 1]$ determined by the numbers $i\alpha \mod 1$, $i = 0, \ldots, n-1$.

To each such sequence we associate its distance encoding, as done previously. There are finitely many distinct encodings, since they are of length n. Thus we may assume, by taking a subsequence of the α_k, that the distance encoding is the same for every k.

Note that an equality of distances for α_k will pass to the limit; this implies that there are no more distances for α than for α_k.

We obtain that there are at most three distances for α, since this is true for the rational numbers α_k, and this proves the three-distance theorem for α irrational.

We assume now that there are exactly three distances for α. Since there are no more than three distances for the α_k, there must be exactly three, and the distance encoding of α is the same as that of each α_k; it is the word encoding of some discrete exchange of three intervals, by the first part of the proof. Moreover, the longest distance for α_k is the sum of the other two, and this passes to the limit too. This ends the proof of the theorem.

It remains only to prove the following lemma.

Lemma 2. *Let $\omega \in S_N$ be a circular discrete exchange of two intervals associated with the composition (p, q) with*

$N = p + q$. Then its cyclic restriction to any $[\![n]\!]$, $n < N$, is a circular discrete exchange of two or three intervals.

Proof. Since ω is circular, σ is circular too, as was noted in part A.1 above. For the same reason, p, q are relatively prime; in particular, $p \neq q$.

Suppose first that $p < q$. Assume that $q < n < N$. Then it is readily verified that σ sends increasingly the intervals $\{0, \ldots, n-q-1\}$, $\{n-q, \ldots, p-1\}$, $\{p, \ldots, n-1\}$ respectively onto the intervals $\{q, \ldots, n-1\}$, $\{n-p, \ldots, q-1\}$, $\{0, \ldots, n-p-1\}$; hence σ is a circular discrete exchange of three intervals associated with the composition $(n-q, N-n, n-p)$ of n. If $n = q$, then σ sends increasingly the intervals $\{1, \ldots, p-1\}$, $\{p, \ldots, n-1\}$ respectively onto the intervals $\{n-p, \ldots, n-1\}$, $\{0, \ldots, n-p-1\}$; hence σ is a circular discrete exchange of two intervals associated with the composition $(p, n-p)$ of n. Now, if $n < q$, then σ is the cyclic restriction to $[\![n]\!]$ of ω', where ω' is the cyclic restriction of ω to $[\![q]\!]$; by the previous argument, ω' is a circular discrete exchange of two intervals, so that by induction on N, σ is a circular discrete exchange of two or three intervals. The case $p > q$ is similar. □

A Dynamical Viewpoint

Let us revisit the previous notions in dynamical terms by considering the dynamical system (with discrete time) acting on $[\![n]\!]$ defined by the permutation σ. Dynamical systems describe the evolution of systems over time. A discrete-time dynamical system (X, T) consists of a phase space X and a map T that acts on it and that governs the discrete-time evolution of elements in X. We then consider the orbits $(x, T(x), T^2(x), \ldots, T^n(x), \ldots)$ of elements of x. Usually the space X is a compact metric space endowed with some probability measure, and ergodic theory allows the description of the long-range statistical behavior of ergodic dynamical systems (see, for instance, [24]).

We consider here maps (permutations) acting on finite sets. In fact, we work with the dynamical system $([\![n]\!], \sigma)$, where σ is a discrete interval exchange. Our tools are purely combinatorial. However, they have continuous and dynamical counterparts when expressed in terms of (continuous) interval exchange transformations. Such dynamical systems generalize circle rotation, i.e., maps of the form $x \mapsto x + \alpha$ modulo 1. For interval exchange transformations, the phase space X is the unit interval $[0, 1]$ divided into a finite number of subintervals, and the transformation T acts by translation by permuting these subintervals.

When we work with the dynamical system $([\![n]\!], \sigma)$, the word encoding is an ℓ-letter word of length n that codes the orbit of 0 under the action of the map σ with respect to the partition of $[\![n]\!]$ by the intervals $(I_j)_{j=1,\ldots,\ell}$. Circularity, i.e., the fact that the orbit $\{0, \sigma(0), \ldots\}$ of 0 visits every element of $[\![n]\!]$, corresponds to the classical notion of minimality in the continuous setting. The cyclic restriction σ of ω to $[\![n]\!]$ is a first return map: ω acts on $[\![N]\!]$, and σ is the permutation encoding the first returns of ω to the subset $[\![n]\!]$.

Lemma 2 is then a discrete reformulation of a classical statement on the induction (that is, on specific choices of first return maps) for continuous interval exchanges; see, e.g., [21]. Note that induction is a basic tool in the study of interval exchanges that generalizes the Euclidean algorithm and continued fractions. The study of interval exchanges is a particularly active and rich subject, which arises, for instance, in the study of polygonal billiards and translation surfaces, such as illustrated by the survey [25].

Perfectly Clustering Words

We have seen with Theorem 1 that the words encoding the successive lengths have a description in terms of circular symmetric discrete interval exchanges. In this final section, we give without proof a further remarkable characterization of the word encodings of circular symmetric discrete interval exchanges. For more on this relation, see [4].

The Burrows–Wheeler transform is a mapping BW from words onto words (of the same length) defined as follows: let v be a word on a totally ordered alphabet. Consider the matrix whose rows are the conjugates[2] of v, ordered lexicographically, with the smallest in the first row (each entry of the matrix is a letter). Then $\mathrm{BW}(v)$ is the word read on the last column of this matrix from the first row to the last. This transform has been widely studied, in particular in information theory for data and text compression.

A word is called perfectly clustering if its image under BW is a decreasing word. Take as an example the word *acacbbc* of the previous sections on the ordered alphabet $\{a < b < c\}$. Its Burrows–Wheeler matrix is by definition

$$\begin{matrix} a & c & a & c & b & b & c \\ a & c & b & b & c & a & c \\ b & b & c & a & c & a & c \\ b & c & a & c & a & c & b \\ c & a & c & a & c & b & b \\ c & a & c & b & b & c & a \\ c & b & b & c & a & c & a \end{matrix}.$$

Thus $\mathrm{BW}(acacbbc) = cccbbaa$, and *acacbbc* is therefore perfectly clustering. One recognizes here the interval exchange of our running example when we compare the first and last columns of the Burrows–Wheeler matrix.

Perfectly clustering words have a further beautiful description. We first need a definition. A word w is said to be a Lyndon word if for each proper factorization $w = uv$, w is smaller than vu for the lexicographical order; equivalently, w of length n is a Lyndon word if it is the first row of its Burrow–Wheeler matrix and if the latter has n rows. For more on Lyndon words, see, e.g., [10] or [14].

We now can relate perfectly clustering words and word encodings of interval exchanges.

[2] Two words are conjugate if they may be written uw and wu for some words u, w; for example, *acacbbc* and *cbbcaca* are conjugate.

Theorem 3. *A Lyndon word is perfectly clustering if and only if it is the word encoding of some circular symmetric discrete interval exchange.*

This theorem is essentially due to Ferenczi and Zamboni [4] (the case of a two-letter alphabet was proved earlier by Mantaci, Restivo, and Sciortino [12]); see [3] for a direct proof.

Thus the distance-encoding word in Theorem 1 is a perfectly clustering Lyndon word.

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