

ON DESCENT ALGEBRAS AND TWISTED BIALGEBRAS

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À Pierre Cartier, en témoignage de notre affection et de notre admiration

ABSTRACT. Bialgebras in the category of tensor species (twisted bialgebras) deserve a particular attention, in particular in view of applications to algebraic combinatorics. In order to study these bialgebras, a new class of descent algebras is introduced. The fine structure of Barratt's permutation bi-ring (the direct sum of the symmetric group algebras) is investigated in detail from this point of view, leading to the definition of an enveloping algebra structure on it.

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1. INTRODUCTION

Many phenomena in algebraic combinatorics can be explained, and sometimes are discovered, using a suitable bialgebra structure. This general fact is particularly true of the combinatorics of the symmetric group. It was emphasized already in 1976, by Ladnor Geissinger, in the setting of the theory of symmetric functions [2], and has been, since then, one of the leading themes in the development of this field of combinatorics. Examples and references on the subject until the beginning of the 90's are contained in [13]. During the last ten years, the subject has evolved. Classical objects, such as the direct sum of the symmetric group algebras, have been provided with bialgebra structures [7], whereas new algebras related to symmetric groups have appeared [3], [5], [12], [11]. Once again, most often these algebras have a bialgebra structure.

The aim of this article is to study systematically and as explicitly as possible another kind of algebraic structure that appears to be as fundamental as classical bialgebras, both for algebraic combinatorics and for its applications to other fields, in particular to algebraic topology. In fact, we are interested in bialgebras in the (symmetric monoidal) category \mathbf{Sp} of *tensor species*. Associative and Lie algebras in \mathbf{Sp} were first introduced by Barratt in order to understand Hopf invariants [1]. He called them, respectively, twisted algebras and twisted Lie algebras. The general

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formalism for dealing with algebraic objects in \mathbf{Sp} (twisted algebras) was introduced by A. Joyal. He has established a Poincaré–Birkhoff–Witt theorem for enveloping algebras in this setting [4]. More recently, it has been proved by C. R. Stover that a Cartier–Milnor–Moore theorem, i. e., an equivalence of categories between connected cocommutative twisted Hopf algebras and connected twisted Lie algebras, also holds [14].

We are interested here in studying the internal structure of particular bialgebras in \mathbf{Sp} . Surprisingly enough, in spite of the seminal article by A. Joyal on the subject, no investigations of this kind have been undertaken, to the best of our knowledge, even for the two fundamental examples of the theory, namely, the tensor algebra and the permutation bi-ring \mathbf{S} (the direct sum of the symmetric group algebras). However, the same phenomenon can be observed with usual bialgebras: their general structural properties (the Cartier–Milnor–Moore theorem, the Leray theorem, ...) were known for a long time when the investigation of the properties of particular bialgebras meaningful for algebraic combinatorics (the tensor algebra, poset Hopf algebras, ...) seriously began.

In order to study the bialgebras in \mathbf{Sp} we introduce a new class of descent algebras, extending to tensor species the constructions in [10]. Besides computing explicitly the “structure constants” of certain twisted bialgebras, giving explicit formulas for coproducts, for some canonical endomorphisms, exhibiting various identities, etc., we show how these structures are related to the corresponding structures in the classical case. For example, we show that the bialgebra structure on \mathbf{S} defined in [7] corresponds naturally to the Joyal twisted enveloping algebra structure on \mathbf{S} . We also deduce from the general properties of twisted bialgebras the existence of a new cocommutative bialgebra structure (and, therefore, an enveloping algebra structure) on \mathbf{S} .

2. TENSOR SPECIES

In this section, we recall briefly some classical definitions [1], [4], [14]. There are two point of views on the theory of algebras in \mathbf{Sp} , that is, on the theory of *twisted algebras*. The first one is Barratt’s: he defines algebras by means of operations and identities involving symmetric group actions. The second one, more conceptual, due to Joyal, consists in defining them as algebras in the symmetric monoidal category of tensor species. Both point of view are equivalent and both are useful in practice, and we therefore mention, for each object, the two possible definitions. In general, having in view applications to computations in the symmetric group algebras, we have tried to give tractable definitions of twisted structures and, as far as possible, explicit formulas.

We write \mathbf{B} for the category of finite sets, where the morphisms are the bijections of sets. We write \mathbf{B}^+ for the *linearization* of \mathbf{B} , that is for the category whose objects are the free abelian groups $\mathbb{Z}S$ over finite sets S and whose morphisms are formal linear combinations (over the integers) of morphisms in \mathbf{B} . For example, $\text{Aut}_{\mathbf{B}^+}(\mathbb{Z}[n]) \cong \mathbb{Z}S_n$, the group algebra of the symmetric group S_n of order n , where we write $[n]$ for the set $\{1, \dots, n\}$. We represent a permutation $\sigma \in S_n$ by the sequence of its values: $\sigma = (\sigma(1), \dots, \sigma(n))$. In the Barratt point of view, \mathbf{B}^+

is replaced by a skeleton, \mathbf{S} . Recall (see, e. g., [6]) that, in any category \mathbf{C} , a *skeleton* of \mathbf{C} is any full subcategory \mathbf{A} such that each object of \mathbf{C} is isomorphic to exactly one object of \mathbf{A} . In the particular case of \mathbf{B}^+ , we choose as objects of the skeleton \mathbf{S} the abelian groups $\mathbb{Z}[n]$. The only morphisms in \mathbf{S} are the automorphisms of the objects, that is the elements of $\text{Aut}_{\mathbf{B}^+}(\mathbb{Z}[n]) = \text{Aut}_{\mathbf{S}}(\mathbb{Z}[n]) \cong \mathbb{Z}S_n$. One of the advantages of the point of view of Barratt is that \mathbf{S} can be viewed as carrying a simple algebraic structure. Recall that a bi-ring is a graded ring $(\Lambda = \bigoplus_{n \in \mathbb{N}} \Lambda_n, \times)$, together with a ring structure (Λ_n, \cdot) on each Λ_n , the products \cdot satisfying the consistency condition [1]:

$$(a \cdot c) \times (b \cdot d) = (a \times b) \cdot (c \times d),$$

when a and c (resp. b and d) have the same degree.

Definition 1. The *permutation bi-ring* \mathbf{S} is, as a graded abelian group, the direct sum $\bigoplus_{n \in \mathbb{N}} \mathbb{Z}S_n$. The product \times is induced by the obvious embeddings $S_n \times S_m \subset S_{n+m}$. The products \cdot are the internal products in the group algebras.

Notice that the products \cdot on the graded components $\mathbb{Z}S_n$ can be extended to \mathbf{S} by requiring that $\sigma \cdot \beta := 0$ if $\sigma \in S_n, \beta \in S_m, n \neq m$. However, \mathbf{S} is then a ring without a unit for this product.

Definition 2. The category of *tensor species* \mathbf{Sp} is the category of functors $\mathbf{Sp} := \mathbf{Ab}^{\mathbf{B}}$ where \mathbf{Ab} is the category of abelian groups.

Tensor species may also be defined over an arbitrary commutative ring R or over a commutative field k . We write \mathbf{Sp}_R (resp. \mathbf{Sp}_k) for $\mathbf{Mod}^{\mathbf{B}}$ (resp. $\mathbf{Vect}^{\mathbf{B}}$), where \mathbf{Mod} is the category of R -modules and \mathbf{Vect} the category of vector spaces over k .

Tensor species correspond to functors $\mathbf{Ab}^{\mathbf{S}}$ or, equivalently to \mathbf{S} -modules, that is graded abelian groups $(M_n)_{n \in \mathbb{N}}$, where each M_n is provided with the structure of a right S_n -module (we follow Barratt and Stover and prefer to consider right instead of left modules). If M is a right S_n -module, $m \in M$ and $\sigma \in S_n$, we write $\sigma(m)$ for $m \cdot \sigma$. In particular, $\beta \circ \sigma(m) = (m \cdot \sigma) \cdot \beta = m \cdot (\sigma \cdot \beta) = \sigma \cdot \beta(m)$. The category of tensor species has direct sums that are induced by pointwise direct sums of abelian groups:

$$\forall F \in \mathbf{Sp}, \forall G \in \mathbf{Sp}, \forall S \in \mathbf{B}, (F \oplus G)[S] := F[S] \oplus G[S],$$

and is symmetric monoidal for the bilinear tensor product:

$$(F \otimes G)[S] := \bigoplus_{A \sqcup B = S} F[A] \otimes G[B],$$

where $A \sqcup B$ is the disjoint union of A and B . The tensor product in the category $\mathbf{S}\text{-Mod}$ of \mathbf{S} -modules is defined accordingly and is also written \otimes :

$$(M \otimes N)_n = \bigoplus_{A \sqcup B = [n]} M_{|A|} \otimes N_{|B|},$$

where $|A|$ is the number of elements in A . Equivalently:

$$(M \otimes N)_n = \bigoplus_{p+q=n} (M_p \otimes N_q) \otimes_{S_p \times S_q} \mathbb{Z}S_n,$$

and since any element σ in S_n can be written uniquely:

$$\sigma = (\alpha \times \beta) \cdot \omega,$$

where ω is a (p, q) -shuffle, that is a permutation such that $\omega^{-1}(1) < \dots < \omega^{-1}(p)$, $\omega^{-1}(p+1) < \dots < \omega^{-1}(p+q)$, any element in $(M \otimes N)_n$ can be rewritten as a linear combination of elements $(x \otimes y) \cdot \omega$, where $x \in M_p$, $y \in M_q$, $p+q=n$ and ω is a (p, q) -shuffle. Here, we write simply $(x \otimes y) \cdot \omega$ for the class of $(x \otimes y) \otimes \omega$ in $(M_p \otimes N_q) \otimes_{S_p \times S_q} \mathbb{Z}S_n$. With this notation, the component $M_{|A|} \otimes N_{|B|}$, $A = \{a_1, \dots, a_p\}$, $a_1 < \dots < a_p$, $B = \{b_1, \dots, b_q\}$, $b_1 < \dots < b_q$ in $(M \otimes N)_n$ identifies with $M_p \otimes N_q \cdot \omega$, where $\omega^{-1}(1) = a_1, \dots, \omega^{-1}(p) = a_p, \omega^{-1}(p+1) = b_1, \dots, \omega^{-1}(n) = b_q$. In particular, the (monoidal) symmetry

$$M_{|A|} \otimes N_{|B|} \xrightarrow{T} N_{|B|} \otimes M_{|A|}$$

is induced by the composition with $T_{q,p}$, where $T_{q,p} = (q+1, \dots, q+p, 1, \dots, q)$:

$$T(m \otimes n \cdot \omega) := n \otimes m \cdot (T_{q,p} \cdot \omega).$$

Indeed, we have:

$$(T_{q,p} \cdot \omega)^{-1}(1) = \omega^{-1}(p+1) = b_1 < \dots < (T_{q,p} \cdot \omega)^{-1}(q) = b_q$$

and

$$(T_{q,p} \cdot \omega)^{-1}(q+1) = \omega^{-1}(1) = a_1 < \dots < (T_{q,p} \cdot \omega)^{-1}(p+q) = a_p$$

and the symmetry isomorphism is S_n -equivariant since for any $\sigma \in S_n$, rewriting $\omega \cdot \sigma$ as $(\alpha \times \beta) \cdot \omega'$, where ω' is a (p, q) -shuffle, we have:

$$\begin{aligned} T((m \otimes n \cdot \omega) \cdot \sigma) &= T(m\alpha \otimes n\beta \cdot \omega') = n\beta \otimes m\alpha \cdot (T_{q,p} \cdot \omega') \\ &= n \otimes m \cdot ((\beta \times \alpha) \cdot T_{q,p} \cdot \omega') = n \otimes m \cdot T_{q,p} \cdot (\alpha \times \beta) \cdot \omega' \\ &= (n \otimes m \cdot (T_{q,p} \cdot \omega)) \cdot \sigma. \end{aligned}$$

We will use frequently the same notation for an \mathbf{S} -module A and for the corresponding tensor species. Notice that we use the same notation for the usual tensor product of ungraded abelian groups and for the tensor product in \mathbf{Sp} and $\mathbf{S}\text{-Mod}$. We will also consider sometimes tensor products in the category of graded abelian groups and will then use the notation $\bar{\otimes}$. That is, if A and B are graded abelian groups: $(A \bar{\otimes} B)_n := \bigoplus_{p+q=n} A_p \otimes B_q$.

Definition 3. A *twisted algebra* is an algebra in the symmetric monoidal category of tensor species.

This definition is due to Joyal [4, Def. 3]. Concretely, a twisted algebra is a tensor species F provided with a product map (which is a map of tensor species): $F \otimes F \rightarrow F$. Associative algebras, commutative algebras, Lie algebras, and so on, are defined accordingly. For example, a twisted associative algebra with unit in \mathbf{Sp} is a tensor species A provided with a product map in \mathbf{Sp} :

$$A \otimes A \xrightarrow{m} A$$

such that associativity holds: $m \circ (m \otimes A) = m \circ (A \otimes m)$. Notice that we write A for the identity morphism of A . The unit condition is defined in the same way: if

we write \mathbb{Z} for the tensor species whose unique non-trivial component is $\mathbb{Z}[\emptyset] := \mathbb{Z}$, then A has to be provided with a unit map $u: \mathbb{Z} \rightarrow A$ satisfying the usual identities.

The same definition applies to twisted coalgebras, twisted bialgebras, and so on: just replace in the usual definitions the category of vector spaces or modules by the category of tensor species. In the following, all associative and twisted associative algebras have a unit, and all coassociative coalgebras and twisted coassociative coalgebras have a counit. Morphisms preserve units and counits.

Equivalently, from the Barratt point of view, a twisted associative algebra is an \mathbf{S} -module A provided with the structure of a graded associative algebra with product $\overline{m}: A \overline{\otimes} A \rightarrow A$ such that the restrictions $\overline{m}_{p,q}: A_p \otimes A_q \rightarrow A_{p+q}$ are $S_p \times S_q$ -equivariant. The corresponding product m from $A \otimes A$ to A is given, for $x \in A_p, y \in A_q$ and ω a (p, q) -shuffle, by:

$$m(x \otimes y \cdot \omega) := \overline{m}_{p,q}(x, y) \cdot \omega.$$

The product commutes with the symmetry of the monoidal structure if and only if

$$m(x \otimes y \cdot \omega) = m(y \otimes x \cdot (T_{q,p} \cdot \omega)),$$

that is, if

$$\overline{m}_{p,q}(x, y) = \overline{m}_{q,p}(y, x) \cdot T_{q,p}.$$

In particular, a twisted associative algebra is commutative whenever $\overline{m}_{p,q}(x, y) = \overline{m}_{q,p}(y, x) \cdot T_{q,p}$. Dually, a twisted coassociative coalgebra is an \mathbf{S} -module A provided with the structure of a graded coassociative coalgebra with coproduct $\overline{\delta}: A \rightarrow A \overline{\otimes} A$ such that the restrictions $\overline{\delta}_{p,q}: A_{p+q} \rightarrow A_p \otimes A_q$ are $S_p \times S_q$ -equivariant. The coproduct is cocommutative if and only if

$$\overline{\delta}_{p,q} = t \circ \overline{\delta}_{q,p} \circ T_{p,q},$$

where t is the canonical twisting operator for tensor products in the category of vector spaces $t: V \otimes W \rightarrow W \otimes V$. If m (resp. δ) is a product (resp. a coproduct) on the tensor species A , we call respectively restricted product and restricted coproduct the corresponding product and coproduct \overline{m} and $\overline{\delta}$ on A viewed as an \mathbf{S} -module.

The definition of a twisted bialgebra, obvious from the Joyal point of view, requires taking carefully into account the equivariance conditions in the Barratt point of view, see, e. g., [14]. That is, a twisted bialgebra is an \mathbf{S} -module A provided with the structure of a twisted associative algebra and of a twisted coassociative coalgebra, such that for all p, q, r, s , the following identity holds:

$$(\overline{m}_{p,r} \otimes \overline{m}_{q,s}) \circ (A_p \otimes t \otimes A_s) \circ (\overline{\delta}_{p,q} \otimes \overline{\delta}_{r,s}) = \overline{\delta}_{p+r,q+s} \circ (S_p \times T_{q,r} \times S_s) \circ \overline{m}_{p+q,r+s},$$

where the action of $S_p \times T_{q,r} \times S_s$ on $A_{p+q+r+s}$ is induced by the right $S_{p+q+r+s}$ -module structure on $A_{p+q+r+s}$. The unit and the counit should be respectively maps of twisted coalgebras and twisted algebras.

There are several fundamental examples of twisted bialgebras. The first one is the twisted tensor bialgebra $Tw(V)$ over a vector space or a free R -module V . As a twisted algebra, it is the free twisted algebra over V . The coproduct is induced (by adjunction) by the diagonal map in the category of vector spaces or R -modules. We will be merely interested in the particular case when V is a copy of \mathbb{Z} . Then,

$Tw(\mathbb{Z})$ identifies, as a twisted algebra, with the bi-ring \mathbf{S} ([1, Def. 3], see the last section for details).

Another important example is the usual tensor algebra $T(V)$ over V . Linearly, following the Barratt point of view, $T(V)$ is the direct sum $\bigoplus_{n \in \mathbb{N}} V^{\otimes n}$. The elements of $V^{\otimes n}$ are linear combinations of tensor products of elements of V that we write using the word notation: $v_1 \dots v_n := v_1 \otimes \dots \otimes v_n$. There is a right action of the symmetric group S_n on $V^{\otimes n}$, defined by:

$$\forall \sigma \in S_n, v_1 \dots v_n \cdot \sigma := v_{\sigma(1)} \dots v_{\sigma(n)}.$$

The twisted associative product written \overline{m} or \cdot is the concatenation product:

$$v_1 \dots v_n \cdot u_1 \dots u_m := v_1 \dots v_n u_1 \dots u_m.$$

It is equivariant with respect to the symmetric group actions:

$$\forall \sigma \in S_n, \forall \beta \in S_m, (v_1 \dots v_n \cdot \sigma) \cdot (u_1 \dots u_m \cdot \beta) = v_1 \dots v_n u_1 \dots u_m \cdot (\sigma \times \beta)$$

and commutative:

$$v_1 \dots v_n \cdot u_1 \dots u_m = (u_1 \dots u_m \cdot v_1 \dots v_n) \cdot T_{m,n}.$$

The unit of the product is the empty word. The coproduct $\overline{\delta}$ is the deconcatenation coproduct:

$$\overline{\delta}_{n,m}(v_1 \dots v_n u_1 \dots u_m) := v_1 \dots v_n \otimes u_1 \dots u_m.$$

It is also equivariant with respect to the symmetric group actions:

$$\begin{aligned} \forall \sigma \in S_n, \forall \beta \in S_m, (\overline{\delta}_{n,m}(v_1 \dots v_n u_1 \dots u_m)) \cdot (\sigma \otimes \beta) \\ = \overline{\delta}_{n,m}(v_1 \dots v_n u_1 \dots u_m \cdot (\sigma \times \beta)). \end{aligned}$$

It is cocommutative:

$$\delta_{n,m}(v_1 \dots v_n u_1 \dots u_m) = v_1 \dots v_n \otimes u_1 \dots u_m = t \circ \overline{\delta}_{m,n} \circ T_{n,m}(v_1 \dots v_n u_1 \dots u_m).$$

The compatibility of the product with the coproduct reads:

$$\begin{aligned} (\overline{m}_{p,r} \otimes \overline{m}_{q,s}) \circ (V^{\otimes p} \otimes t \otimes V^{\otimes s}) \circ (\overline{\delta}_{p,q} \otimes \overline{\delta}_{r,s})(u_1 \dots u_p v_1 \dots v_q \otimes w_1 \dots w_r z_1 \dots z_s) \\ = (\overline{m}_{p,r} \otimes \overline{m}_{q,s}) \circ (V^{\otimes p} \otimes t \otimes V^{\otimes s})(u_1 \dots u_p \otimes v_1 \dots v_q \otimes w_1 \dots w_r \otimes z_1 \dots z_s) \\ = \overline{m}_{p,r} \otimes \overline{m}_{q,s}(u_1 \dots u_p \otimes w_1 \dots w_r \otimes v_1 \dots v_q \otimes z_1 \dots z_s) \\ = u_1 \dots u_p w_1 \dots w_r \otimes v_1 \dots v_q z_1 \dots z_s \\ = \overline{\delta}_{p+r,q+s}(u_1 \dots u_p w_1 \dots w_r v_1 \dots v_q z_1 \dots z_s) \\ = \overline{\delta}_{p+r,q+s} \circ (S_p \times T_{q,r} \times S_s)(u_1 \dots u_p v_1 \dots v_q w_1 \dots w_r z_1 \dots z_s) \\ = \overline{\delta}_{p+r,q+s} \circ (S_p \times T_{q,r} \times S_s) \circ \overline{m}_{p+q,r+s}(u_1 \dots u_p v_1 \dots v_q \otimes w_1 \dots w_r z_1 \dots z_s). \end{aligned}$$

Finally, another example of twisted bialgebras is provided by twisted enveloping algebras of twisted Lie algebras [4], [14]. In fact, according to Barratt's seminal ideas on the subject, twisted Lie algebras should be an important tool for understanding Hopf invariants: "several forces have made me take up again the notion of homotopy envelopes, where the milling crowd of generalized Hopf invariants may be reduced or at least quieted. The first expository step is the description of twisted Lie algebras..." [1].

3. DESCENT ALGEBRAS OF TWISTED BIALGEBRAS

To a graded bialgebra is canonically associated a descent algebra. Recall its definition [10]. Let B be a graded bialgebra over \mathbb{Z} with product m , coproduct δ , unit ϵ and counit η . The convolution product $*$ of linear endomorphisms of B :

$$\forall (f, g) \in \text{End}(B), \quad f * g := m \circ (f \otimes g) \circ \delta$$

defines the structure of an associative algebra (with unit $\epsilon \circ \eta$) on $\text{End}(B)$. The descent algebra $\text{Desc}(B)$ of B is the subalgebra generated by the projections on the graded components of B : $\pi_n: B \rightarrow B_n$. Moreover, when B is connected ($B_0 = \mathbb{Z}$) and either commutative or cocommutative, it can be shown that the descent algebra is a quotient of Solomon's descent algebra or of the dual algebra.

We are going to study in this section the descent algebras that can be associated to twisted algebras by the same process. We call \mathbf{S} -linear endomorphisms of a twisted bialgebra A the endomorphisms of A viewed as an element of \mathbf{Sp} or $\mathbf{S}\text{-Mod}$. First of all, the convolution product of \mathbf{S} -linear endomorphisms of A can be defined in the same way as in the classical case and inherits the properties of the usual convolution product of endomorphisms of bialgebras. This is a general fact, that holds for bialgebras in any linear symmetric monoidal category.

Definition 4. The *convolution product* of two \mathbf{S} -linear endomorphisms f, g of a twisted bialgebra A with product m , coproduct δ , unit ϵ and counit η is, by definition, the \mathbf{S} -linear endomorphism: $f * g := m \circ (f \otimes g) \circ \delta$.

Proposition 5. *The convolution product provides the abelian group $\text{End}_{\mathbf{S}}(A)$ of \mathbf{S} -linear endomorphisms of A with the structure of an associative algebra with unit $\iota = \epsilon \circ \eta$.*

If A is a tensor species, we write A_n for the sub-tensor species of A defined by: $A_n[S] = 0$ if $|S| \neq n$, $A_n[S] = A[S]$ otherwise.

Definition 6. The *descent algebra* $\text{Desc}(A)$ of a twisted bialgebra is the convolution subalgebra of $\text{End}_{\mathbf{S}}(A)$ generated by the canonical projections, $\pi_n: A \rightarrow A_n$.

Since any twisted bialgebra is naturally an associative algebra for the associated product \bar{m} and a coassociative coalgebra for the associated coproduct $\bar{\delta}$ (but not a bialgebra, in general), we would like to relate first the two convolution products that can be associated naturally to A . We write $\bar{*}$ for the convolution product defined using the underlying non twisted algebraic structures, that is:

$$f \bar{*} g := \bar{m} \circ (f \bar{\otimes} g) \circ \bar{\delta},$$

and we call $\bar{*}$ the *reduced convolution product*. The reduced convolution product provides the set $\text{End}(A)$ of linear endomorphisms of A with the structure of an associative algebra with unit. One of the surprising features of the theory of twisted bialgebras is that the convolution product on $\text{Desc}(A)$ can be related explicitly to the reduced convolution product using the partitions of the symmetric group into descent classes and the corresponding Solomon's elements in the group algebras.

Let us compute explicitly the convolution product on $\text{End}_{\mathbf{S}} A$. By definition, $(A \otimes A)_n := \bigoplus_{S \sqcup T} A_{|S|} \otimes A_{|T|}$, where the sum is over all partitions of $[n]$ into two

disjoint subsets. Recall also that, if we write as an increasing sequence the elements s_i of S followed by the elements t_i of T , the corresponding permutation $\sigma_{S,T} = (s_1, \dots, s_{|S|}, t_1, \dots, t_{|T|})$ in S_n is the inverse of a $(|S|, |T|)$ -shuffle. Moreover, the direct summand $A_{|S|} \otimes A_{|T|}$ of $(A \otimes A)_n$ can be identified with the direct summand $A_p \otimes A_q \cdot \sigma_{S,T}^{-1}$ of $(A_p \otimes A_q) \otimes_{S_p \times S_q} \mathbb{Z}S_n$ if $|S| = p, |T| = q$.

The restriction to A_n of the diagonal map $\delta: A \rightarrow A \otimes A$ decomposes into a sum of linear maps: $\delta_{S,T}: A_n \rightarrow A_{|S|} \otimes A_{|T|}$. If $S = [|S|]$ and $T = \{|S| + 1, \dots, |S| + |T|\}$, $\delta_{S,T} = \bar{\delta}_{|S|,|T|}$. The other components can be recovered from the knowledge of $\bar{\delta}_{|S|,|T|}$ using the equivariance properties of δ . Namely, we have: $\delta_{S,T} = \sigma_{S,T}^{-1} \circ \bar{\delta}_{|S|,|T|} \circ \sigma_{S,T}$. Therefore, for f, g in $\text{End}_{\mathbf{S}}(A)$ with f of degree r and g of degree s , that is: $f = \pi_r \circ f, g = \pi_s \circ g$, we have, since the product and the tensor product in $\mathbf{S}\text{-Mod}$ are equivariant:

$$\begin{aligned} f * g &= m \circ (f \otimes g) \circ \delta \\ &= \sum_{\substack{S \sqcup T = [r+s] \\ |S|=r, |T|=s}} \sigma_{S,T}^{-1} \circ \bar{m}_{r,s} \circ (f \bar{\otimes} g) \circ \bar{\delta}_{r,s} \circ \sigma_{S,T} \\ &= \left(\sum_{\substack{S \sqcup T = [r+s] \\ |S|=r, |T|=s}} \sigma_{S,T} \right) \bullet (f \bar{*} g), \end{aligned}$$

where we write \bullet for the conjugation action of S_n on $\text{End}(A)$ and for its linear extension to $\mathbb{Z}S_n$:

$$\sigma \bullet f(a) := \sigma^{-1} \circ f \circ \sigma(a) = f(a \cdot \sigma) \cdot \sigma^{-1}.$$

Notice that this action is a left action on $\text{End}(A)$. Besides, the element

$$\sum_{\substack{S \sqcup T = [r+s] \\ |S|=r, |T|=s}} \sigma_{S,T}$$

of $\mathbb{Z}S_{r+s}$ is the sum of all permutations having at most one descent, in position r . By definition, it is exactly the Solomon element $D_{\subseteq \{r\}}$, also written $q_{(r,s)}$, of $\mathbb{Z}S_{r+s}$, see [13, sect. 9]. In other terms:

$$f * g = q_{(r,s)} \bullet (f \bar{*} g).$$

These computations can go a step further: let us compute the convolution product of three \mathbf{S} -linear endomorphisms of A of degrees r, s, t ; the general case follows easily by induction. Then,

$$\begin{aligned} (f * g) * h &= q_{(r+s,t)} \bullet ((f * g) \bar{*} h) \\ &= q_{(r+s,t)} \bullet ((q_{(r,s)} \bullet (f \bar{*} g)) \bar{*} h) \\ &= q_{(r+s,t)} \bullet ((q_{(r,s)} \times S_t) \bullet (f \bar{*} g \bar{*} h)) \\ &= q_{(r,s,t)} \bullet (f \bar{*} g \bar{*} h), \end{aligned}$$

where the third identity follows from the equivariance properties of $\delta_{r+s,t}$ and $\pi_{r+s,t}$. The identity $q_{(r+s,t)} \bullet (q_{(r,s)} \times S_t) = q_{(r,s,t)}$ follows from the properties of descent

elements and can be checked by a direct computation or deduced from the general algebraic properties of shuffles.

Theorem 7. *The convolution product and the reduced convolution product are related by the following formulas: if f_1, \dots, f_r are \mathbf{S} -linear endomorphisms of A of degrees n_1, \dots, n_r :*

$$f_1 * \dots * f_n = q_{(n_1, \dots, n_r)} \bullet (f_1 \bar{*} \dots \bar{*} f_n).$$

This theorem is the fundamental reason why the explicit computations undertaken in [14] in order to prove the Poincaré–Birkhoff–Witt theorem and the Cartier–Milnor–Moore theorem for twisted enveloping algebras and twisted bialgebras involve technical computations with shuffles. The proofs of Stover could in fact be simplified, at least over a ground ring containing the rational numbers, by adapting *mutatis mutandis* the convolution algebra proofs of the structure theorems for classical bialgebras in [10] to the case of twisted bialgebras.

Let us recall, as a conclusion of this section, a construction due to Stover [14, Remark 14.7] relating twisted bialgebras to classical bialgebras, that is, the construction of a natural map from twisted bialgebras to bialgebras. Let us write $\hat{\delta}_{r,s}$ for the composition:

$$A \xrightarrow{\hat{\delta}} \bigoplus_{\substack{S \sqcup T = [r+s] \\ |S|=r, |T|=s}} A_{|S|} \otimes A_{|T|} \xrightarrow{\zeta} A_r \otimes A_s,$$

where ζ is the linear map induced by the canonical isomorphisms $A_{|S|} \otimes A_{|T|} \cong A_r \otimes A_s$. It follows immediately from our previous computations that

$$\hat{\delta}_{r,s} = \bar{\delta}_{r,s} \circ q_{(r,s)}.$$

Therefore, if $t + u = s$ and since $\delta_{r,s}$ is $S_r \times S_t$ -equivariant:

$$\begin{aligned} (A_r \otimes \hat{\delta}_{t,u}) \circ \hat{\delta}_{r,s} &= (A_r \otimes \bar{\delta}_{t,u}) \circ (S_r \times q_{(t,u)}) \circ \bar{\delta}_{r,s} \circ q_{(r,s)} \\ &= (A_r \otimes \bar{\delta}_{t,u}) \circ \bar{\delta}_{r,s} \circ (S_r \times q_{(t,u)}) \circ q_{(r,s)} \\ &= (A_r \otimes \bar{\delta}_{t,u}) \circ \bar{\delta}_{r,s} \circ q_{(r,t,u)} \end{aligned}$$

and, by symmetry:

$$(\hat{\delta}_{r,t} \otimes A_u) \circ \hat{\delta}_{r+t,u} = (\bar{\delta}_{r,t} \otimes A_u) \circ \bar{\delta}_{r+t,u} \circ q_{(r,t,u)}.$$

Since $\bar{\delta}$ defines a coalgebra structure on A , it follows that:

$$(A_r \otimes \hat{\delta}_{t,u}) \circ \hat{\delta}_{r,s} = (\hat{\delta}_{r,t} \otimes A_u) \circ \hat{\delta}_{r+t,u},$$

so that $\hat{\delta} = \sum_{r,s} \hat{\delta}_{r,s}$ defines a coassociative coalgebra structure on A .

Moreover, $\hat{\delta}$ together with \bar{m} define a bialgebra structure on A . This property was asserted *en passant* by Stover without a proof. Because of its usefulness in view of applications to algebraic combinatorics and free Lie algebras computation (see the next sections of this article), we think it necessary to give some details. Let us show that, for $x \in A_p$ and $y \in A_q$:

$$\hat{\delta} \circ \bar{m}(x \otimes y) = (\bar{m} \otimes \bar{m}) \circ (A \otimes t \otimes A) \circ (\hat{\delta} \otimes \hat{\delta})(x \otimes y).$$

We have:

$$\begin{aligned} & (\bar{m} \otimes \bar{m}) \circ (A \bar{\otimes} t \bar{\otimes} A) \circ (\hat{\delta} \bar{\otimes} \hat{\delta})(x \otimes y) \\ &= \sum_{\substack{s+t=p \\ u+v=q}} (\bar{m}_{s,u} \otimes \bar{m}_{t,v}) \circ (A_s \otimes t \otimes A_v) \circ (\bar{\delta}_{s,t} \otimes \bar{\delta}_{u,v}) \circ (q_{(s,t)} \otimes q_{(u,v)})(x \otimes y) \\ &= \sum_{\substack{s+t=p \\ u+v=q}} \bar{\delta}_{s+u,t+v} \circ (S_s \times T_{t,u} \times S_v) \circ \bar{m}_{s+t,u+v} \circ (q_{(s,t)} \otimes q_{(u,v)})(x \otimes y) \end{aligned}$$

and, since $\bar{m}_{s+t,u+v}$ is $S_{s+t} \times S_{u+v}$ -equivariant:

$$= \sum_{\substack{s+t=p \\ u+v=q}} \bar{\delta}_{s+u,t+v} \circ (S_s \times T_{t,u} \times S_v) \circ (q_{(s,t)} \times q_{(u,v)}) \cdot \bar{m}_{s+t,u+v}(x \otimes y),$$

where $q_{(s,t)} \times q_{(u,v)}$ is defined by extending bilinearly the product \times on permutations. We have therefore to prove that, given m and n such that $m + n = p + q$, we have:

$$= \sum_{\substack{s+t=p, u+v=q \\ s+u=m, t+v=n}} (q_{(s,t)} \times q_{(u,v)}) \cdot (S_s \times T_{t,u} \times S_v) = q_{(m,n)}.$$

This amounts to construct a suitable bijection ϕ between the set of all permutations $\sigma \times \beta$, where $\sigma \in S_{s+t}$, $s+t = p$ has at most one descent in position s and $\beta \in S_{u+v}$, $u+v = q$ has at most one descent in position u , and the set of permutations with at most one descent in position $s+u$. We define such a bijection by:

$$\begin{aligned} \phi(\sigma \times \beta) &= (\sigma(1), \dots, \sigma(s), \beta(1) + s + t, \dots, \beta(u) + s + t, \\ &\quad \sigma(s + 1), \dots, \sigma(s + t), \beta(u + 1) + s + t, \dots, \beta(u + v) + s + t). \end{aligned}$$

Definition 8. We call the bialgebra $\hat{A} := (A, \bar{m}, \hat{\delta})$ the *cosymmetrized bialgebra* associated to the twisted bialgebra A .

4. THE TENSOR ALGEBRA

We assume in this section that V is a vector space over a commutative field k with a finite or countable basis $X = (x_1, \dots, x_n, \dots)$. The results in this section also apply if V is an R -module freely generated by x_1, \dots, x_n, \dots , where R is an arbitrary commutative ring (e. g., \mathbb{Z}).

Let us compute explicitly the coproduct on the cosymmetrized bialgebra $\hat{T}(V)$. Let $v_1 \dots v_n \in T(V)$, we have, for all $S = \{s_1, \dots, s_k\}$, $s_1 \leq \dots \leq s_k$, $T = \{t_1, \dots, t_l\}$, $t_1 \leq \dots \leq t_l$, $S \sqcup T = [n]$:

$$\begin{aligned} \sigma_{S,T} \circ \delta_{S,T}(v_1 \dots v_n) &= \bar{\delta}_{k,l}((v_1 \dots v_n) \cdot \sigma_{S,T}) \\ &= \bar{\delta}_{k,l}(v_{s_1} \dots v_{s_k} v_{t_1} \dots v_{t_l}) \\ &= v_{s_1} \dots v_{s_k} \otimes v_{t_1} \dots v_{t_l}. \end{aligned}$$

Therefore:

$$\hat{\delta}(v_1 \dots v_n) = \sum_{S \sqcup T = [n]} v_{s_1} \dots v_{s_k} \otimes v_{t_1} \dots v_{t_l},$$

that is, $\hat{\delta}$ identifies with the *unshuffle coproduct* (the dual of the shuffle product for the scalar product on $T(V)$ for which the words in the letters of X form an orthonormal basis).

Proposition 9. *The cosymmetrization of the tensor algebra viewed as a twisted bialgebra is the usual tensor bialgebra with product the concatenation product and coproduct the shuffle coproduct.*

It is well known that the primitive part of the tensor algebra over X is the free Lie algebra over X if the characteristic of the ground ring is zero. Equivalently, over a field of characteristic zero, the tensor algebra is canonically isomorphic to the enveloping algebra of the free Lie algebra. The same property holds for twisted algebras, that is: the free twisted tensor algebra over X is the enveloping algebra of the free twisted Lie algebra, see [4], [14]. The next proposition shows that the tensor algebra is the free commutative twisted algebra.

Proposition 10. *Over any commutative ring R , the primitive elements of the tensor algebra $T(V)$, viewed as a twisted bialgebra, are the elements of V . In particular, the tensor algebra $T(V)$ is a free commutative twisted algebra over V .*

Indeed, let $x \in T(V)_n$. If $I = (i_1, \dots, i_n)$, we write x_I for the word $x_{i_1} \dots x_{i_n}$. Then, $x = \sum_{I=(i_1, \dots, i_n)} \alpha_I x_I$, where the number of non zero coefficients α_I is finite. We want to show that, if x is primitive and $n \geq 2$, that is, if $\delta(x) = x \otimes 1 + 1 \otimes x \in T(V) \otimes T(V)$, then $x = 0$ (the elements of V are obviously primitive elements). It is enough to show that, if $x \neq 0$, then $\bar{\delta}_{2,n-2}(x) \neq 0$, a property that follows easily from the definition of the deconcatenation product. Indeed, we have that:

$$\bar{\delta}_{2,n-2}(x) = \sum_{I=(i_1, \dots, i_n)} \alpha_I x_{i_1} x_{i_2} \otimes x_{i_3} \dots x_{i_n}$$

is non zero since $\bar{m} \circ \bar{\delta}_{2,n-2}(x) = \sum_{I=(i_1, \dots, i_n)} \alpha_I x_I = x$.

The last part of the proposition follows: indeed, the set of primitive elements $\text{Prim}(B)$ of a twisted bialgebras B carries naturally the structure of a twisted Lie algebra. As a twisted Lie algebra, $V = \text{Prim}(T(V))$ has a trivial bracket (e.g., since the bracketing is a graded map), and $T(V)$ is therefore, in view of Stover's Cartier–Milnor–Moore theorem [14, Th 8.4], the enveloping algebra of a trivial, i. e., commutative twisted Lie algebra, that is, a free commutative twisted algebra.

For a general bialgebra, we have:

Proposition 11. *The set of primitive elements in the twisted bialgebra A is a subset of the set of primitive elements of the bialgebra \hat{A} .*

The proposition follows from the definition of the coproduct $\hat{\delta}$, whose components $\hat{\delta}_{n,m}$ are 0 if the components $\delta_{S,T}$, $|S| = n$, $|T| = m$, of δ are 0.

5. THE PERMUTATION BI-RING

Recall that the permutation bi-ring $\mathbf{S} := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}S_n$ has a graded product \times . This product provides \mathbf{S} with the structure of a twisted algebra and, in fact, \mathbf{S} can be naturally identified with the free twisted algebra on one generator since

$\mathbf{S} \cong \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[1]^{\otimes n} = Tw(\mathbb{Z})$, where we write $\mathbb{Z}[1]$ for the graded \mathbf{S} -module with only non trivial component \mathbb{Z} , in degree 1. We also write \times for the product map from $\mathbf{S} \otimes \mathbf{S}$ to \mathbf{S} induced by \times . Notice that the restriction of \times to $(\mathbf{S}_n \otimes \mathbf{S}_m) \otimes_{S_n \times S_m} \mathbb{Z}S_{n+m}$ is simply the isomorphism:

$$(\mathbb{Z}S_n \otimes \mathbb{Z}S_m) \otimes_{S_n \times S_m} \mathbb{Z}S_{n+m} \cong \mathbb{Z}S_{n+m}.$$

Since \mathbf{S} is free as a twisted algebra, there is a unique coproduct δ making it a twisted Hopf algebra. It can be computed explicitly as follows. For simplicity, we write 1_n instead of S_n for the unit (the identity map) of S_n . Notice first that the elements of $\mathbb{Z}S_1$ (the degree one elements of the graded vector space \mathbf{S}) have to be primitive elements, and that we have:

$$\delta(1_n) = \delta(1_{n-1} \times 1_1) = (\times \otimes \times) \circ (\mathbf{S} \otimes T \otimes \mathbf{S}) \circ (\delta \otimes \delta)(1_{n-1} \otimes 1_1),$$

where T is the twisting operator (in the category of \mathbf{S} -modules) acting on $\mathbf{S} \otimes \mathbf{S}$.

From now on, if $\gamma = \sum_{\alpha \in S_n} a_\alpha \cdot \alpha \in \mathbb{Z}S_n$, we write γ^* for $\sum_{\alpha \in S_n} a_\alpha \cdot \alpha^{-1}$. In particular, we have:

$$q_{(n,m)}^* = \sum_{\substack{S \sqcup T = [n+m] \\ |S|=n}} \sigma_{S,T}^{-1}.$$

Proposition 12. *The coproduct on \mathbf{S} is cocommutative. It is given by:*

$$\delta(1_n) = \sum_{p+q=n} \sum_{\substack{S \sqcup T = [n] \\ |S|=p}} 1_p \otimes 1_q \cdot \sigma_{S,T}^{-1}$$

and, on an arbitrary element of S_n , by:

$$\delta(\sigma) = \delta(1_n) \cdot \sigma,$$

or, equivalently, by:

$$\delta(\sigma) = \left(\sum_{p+q=n} 1_p \otimes 1_q \cdot q_{(p,q)}^* \right) \cdot \sigma.$$

Let us show that the first formula follows by induction. The second formula follows from the first one by equivariance.

According to the previous computation we have, assuming that the formula holds for $\delta(1_{n-1})$:

$$\begin{aligned} \delta(1_n) &= (\times \otimes \times) \circ (\mathbf{S} \otimes T \otimes \mathbf{S})(\delta(1_{n-1}) \otimes \delta(1_1)) \\ &= (\times \otimes \times) \circ (\mathbf{S} \otimes T \otimes \mathbf{S}) \left[\left(\sum_{p+q=n-1} \sum_{\substack{S \sqcup T = [n-1] \\ |S|=p}} 1_p \otimes 1_q \cdot \sigma_{S,T}^{-1} \right) \otimes (1_1 \otimes 1_0 + 1_0 \otimes 1_1) \right] \\ &= \sum_{p+q=n-1} \sum_{\substack{S \sqcup T = [n-1] \\ |S|=p}} (\times \otimes \times) \circ (\mathbf{S} \otimes T \otimes \mathbf{S}) [1_p \otimes 1_q \otimes 1_1 \otimes 1_0 \cdot (\sigma_{S,T}^{-1} \times S_1 \times S_0) \\ &\quad + 1_p \otimes 1_q \otimes 1_0 \otimes 1_1 \cdot (\sigma_{S,T}^{-1} \times S_0 \times S_1)] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{p+q=n-1} \sum_{\substack{S \sqcup T = [n-1] \\ |S|=p}} (\times \otimes \times) [1_p \otimes 1_1 \otimes 1_q \otimes 1_0 \cdot (S_p \times T_{1,q} \times S_0) \cdot (\sigma_{S,T}^{-1} \times S_1 \times S_0) \\
 &\quad + 1_p \otimes 1_0 \otimes 1_q \otimes 1_1 \cdot (S_p \times T_{0,q} \times S_1) \cdot (\sigma_{S,T}^{-1} \times S_0 \times S_1)];
 \end{aligned}$$

since

$$\begin{aligned}
 &S_p \times T_{1,q} \times S_0 \cdot (\sigma_{S,T}^{-1} \times S_1 \times S_0) \\
 &= (1, \dots, p, p+2, \dots, p+q+1, p+1) \cdot (\sigma_{S,T}^{-1}(1), \dots, \sigma_{S,T}^{-1}(p+q), p+q+1) \\
 &= \sigma_{S'=\mathcal{S} \sqcup \{p+q+1\}, T}^{-1}
 \end{aligned}$$

and $(S_p \times T_{0,q} \times S_1) \cdot (\sigma_{S,T}^{-1} \times S_1 \times S_0) = \sigma_{S,T \sqcup \{p+q+1\}}^{-1}$, we have:

$$\begin{aligned}
 \delta(1_n) &= \sum_{p+q=n-1} \sum_{\substack{S \sqcup T = [n-1] \\ |S|=p}} \left(1_{p+1} \times 1_q \cdot \sigma_{S \sqcup \{n\}, T}^{-1} + 1_p \otimes 1_{q+1} \cdot \sigma_{S, T \sqcup \{n\}}^{-1} \right) \\
 &= \sum_{p'+q'=n} \sum_{\substack{U \sqcup V = [n] \\ |U|=p'}} 1_{p'} \otimes 1_{q'} \cdot \sigma_{U,V}^{-1}.
 \end{aligned}$$

The cocommutativity follows, for example, from the computation of the reduced coproduct, see Proposition 13 below.

To compute the restricted coproduct, let us first recall the definition of the standardization map. The standardization is the map associating to a word of length k , $X = x_1 \dots x_k$ in the letters $1, \dots, n$ and without repetition of the letters, the unique word $\text{st}(X) = y_1 \dots y_k$ in the letters $1, \dots, k$, without repetition of the letters and such that the relative order of the letters is preserved: $y_i \leq y_j$ if and only if $x_i \leq x_j$. For example, $\text{st}(3745) = 1423$; see, e.g., [11] for examples of uses of the standardization map in free Lie algebra computations.

If $\sigma = (\sigma(1), \dots, \sigma(n)) \in S_n$, we write σ_i^- for the word $\sigma(1) \dots \sigma(i)$ and σ_{i+1}^+ for $\sigma(i+1) \dots \sigma(n)$.

Proposition 13. *The restricted coproduct $\bar{\delta}$ is given by:*

$$\forall \sigma \in S_n, \quad \bar{\delta}(\sigma) = \sum_{i=0}^n \text{st}(\sigma_i^-) \otimes \text{st}(\sigma_{i+1}^+).$$

To prove the proposition, we have to identify in $\delta(\sigma)$ the component lying in $\mathbb{Z}S_i \otimes \mathbb{Z}S_{\{i+1, \dots, n\}}$. That is, we have to find out what are the subsets S, T of $[n]$ such that $S \sqcup T = [n]$, $|S| = i$ and $\sigma_{S,T}^{-1} \cdot \sigma \in S_i \times S_{n-i}$. Recall that $\omega = \sigma_{S,T}^{-1}$ is the $(i, n-i)$ -shuffle: $\omega^{-1}(1) = s_1, \dots, \omega^{-1}(i) = s_i, \omega^{-1}(i+1) = t_1, \dots, \omega^{-1}(n) = t_{n-i}$, where s_1, \dots, s_i (resp. t_1, \dots, t_{n-i}) are the elements of S (resp. T) written in increasing order. The conditions are fulfilled if and only if: $\{\sigma^{-1}(s_1), \dots, \sigma^{-1}(s_i)\} = [i]$. We then have:

$$\{\sigma(1), \dots, \sigma(i)\} = \{s_1, \dots, s_i\} = \{\omega^{-1}(1), \dots, \omega^{-1}(i)\}$$

and therefore, $\exists \alpha \in S_i, \sigma(k) = \omega^{-1} \cdot \alpha(k)$ when $k \leq i$, and $\exists \beta \in S_{n-i}, \sigma(k) = \omega^{-1}(i + \beta(k - i))$, when $k > i$. Since the restriction of ω^{-1} to $1, \dots, i$ is an

increasing map to S , α is nothing but the permutation associated to the standardization of $\sigma(1) \dots \sigma(i)$. That is, $\alpha = \text{st}(\sigma_i^-)$. The same argument shows that $\beta = \text{st}(\sigma_{\{i+1, \dots, n\}}) = \text{st}(\sigma_{i+1}^+)$, and the proposition follows.

Corollary 14. *The restricted coproduct is identical to the coproduct on \mathbf{S} defined in [7].*

Indeed, one of the two coproducts associated to the two dual bialgebra structures on \mathbf{S} introduced in [7] is defined by the formula of Proposition 13.

The construction of the cosymmetrization of a twisted bialgebra may be dualized [14, Sect. 14.4]. That is, given a twisted bialgebra A , one may construct a new product on A by setting: $\hat{m}(x \otimes y) := \bar{m}(x \otimes y) \cdot q_{(n,m)}^*$ if $x \in A_n, y \in A_m$. It is associative since $\bar{m}_{n,m}$ is $S_n \times S_m$ -equivariant, since $*$ is an anti-involution, and since

$$q_{(n+m,p)} \cdot (q_{(n,m)} \times S_p) = q_{(n,m,p)} = q_{(n,m+p)} \times (S_n \times q_{(m,p)}).$$

Proposition 15. *The triple $\bar{A} := (A, \hat{m}, \bar{\delta})$ is a (classical) bialgebra. We call it the symmetrized bialgebra associated to the twisted bialgebra A .*

The proof of the Proposition follows by duality from the proof that \bar{m} and $\hat{\delta}$ define a bialgebra structure on A .

Proposition 16. *The symmetrized bialgebra $\bar{\mathbf{S}}$ of \mathbf{S} can be identified with the Hopf algebra structure on \mathbf{S} defined in [7].*

Indeed, we have, for $\sigma \in S_p, \beta \in S_q$:

$$\hat{m}(\sigma, \beta) = (\sigma \times \beta) \cdot q_{(p,q)}^*.$$

In the following, we write $\bar{\delta}^T$ and \bar{m}^T respectively for the reduced coproduct and product on the tensor algebra $T(V)$ (see Section 2). Let us compute the action of $\hat{m}(\sigma, \beta)$ on a word $x_1 \dots x_{p+q}$:

$$\begin{aligned} x_1 \dots x_{p+q} \cdot \hat{m}(\sigma, \beta) &= x_1 \dots x_{p+q} \cdot (\sigma \times \beta) \cdot q_{(p,q)}^* \\ &= q_{(p,q)}^* \circ \bar{m}_{p,q}^T \circ (\sigma \times \beta) \circ \bar{\delta}^T(x_1 \dots x_{p+q}). \end{aligned}$$

According to [7], the product on \mathbf{S} associated to the reduced coproduct $\bar{\delta}$ is the convolution product on \mathbf{S} defined by embedding \mathbf{S} into the set of linear endomorphisms of $T(V)$ provided with the deconcatenation coproduct $\bar{\delta}^T$ and with the shuffle product ω . In our notation:

$$v_1 \dots v_p \omega w_1 \dots w_q := v_1 \dots v_p w_1 \dots w_q \cdot q_{(p,q)}^*,$$

and therefore:

$$v_1 \dots v_p \omega w_1 \dots w_q = q_{(p,q)}^* \circ \bar{m}_{p,q}^T(v_1 \dots v_p \otimes w_1 \dots w_q).$$

The proposition follows.

To conclude the description of \mathbf{S} , it remains, since it is a cocommutative twisted bialgebra, to describe it as an enveloping algebra or, equivalently, to compute its primitive part. Notice that some partial results on the subject were already obtained in [1], [4].

Recall that there is a natural scalar product $\langle | \rangle$ on $\mathbb{Z}S_n$ for which the elements of S_n form an orthonormal basis. Recall also that the Lie elements of $\mathbb{Z}S_n$ are the elements that are sent to elements of the free Lie algebra over $[n]$ when permutations are rewritten as words. For example, the Lie bracket:

$$[[1, 2], 3] = 123 - 213 - 312 + 321$$

is associated to the Lie element

$$(123) - (213) - (312) + (321).$$

The Lie elements of $\mathbb{Z}S_n$ are orthogonal to proper shuffles and are characterized by this property. See [13, Chap. 8] for further details. Assume that $\alpha \in \mathbb{Z}S_n$ is a primitive element for δ . According to Proposition 12, this means that, for any $0 < i < n$, $q_{(i,n-i)}^* \cdot \alpha = 0$. Equivalently, we have:

$$\forall \sigma \in S_n, \langle q_{(i,n-i)}^* \cdot \alpha, \sigma \rangle = 0,$$

or $\langle \sigma^{-1} q_{(i,n-i)}^* \cdot \alpha^*, \alpha^* \rangle = 0$. That is: α^* is orthogonal to all proper shuffles. Therefore, α is a primitive element if and only α^* belongs to Lie_n , the set of Lie elements in $\mathbb{Z}S_n$.

Proposition 17. *The twisted Lie algebra of primitive elements in \mathbf{S} is the direct sum of the images under $\sigma \rightarrow \sigma^{-1}$ of the Lie elements in $\mathbb{Z}S_n$, that is, of Lie_n :*

$$\text{Prim}(\mathbf{S}) = \bigoplus_n \text{Lie}_n^*.$$

Compare with Theorem 12 of [11]: $\bigoplus_n \text{Lie}_n$ also carries naturally a graded Lie algebra structure induced by the convolution product in \mathbf{S} ; its enveloping algebra for this Lie algebra structure is the algebra introduced and studied in [11].

6. HOPF ALGEBRA STRUCTURES ON \mathbf{S}

In this section, the ground ring R is the field \mathbb{Q} , or any ring containing \mathbb{Q} . We have already pointed out that the symmetrization of \mathbf{S} leads to a bialgebra structure on \mathbf{S} that can be identified with the one introduced in [7]. In this section, we describe the cosymmetrization of \mathbf{S} . This leads to a new bialgebra structure $\hat{\mathbf{S}}$ on \mathbf{S} that has the property of being cocommutative. Therefore, since $\hat{\mathbf{S}}$ is also connected, and according to the Cartier–Milnor–Moore theorem [8], [10], $\hat{\mathbf{S}}$ is an enveloping algebra. Moreover, we show that the Lie algebra of primitive elements in $\hat{\mathbf{S}}$ is a free Lie algebra.

Proposition 18. *The cosymmetrized coproduct on \mathbf{S} is given by:*

$$\forall \sigma \in S_n, \hat{\delta}(\sigma) = (\text{st} \otimes \text{st}) \circ \hat{\delta}^T(\sigma).$$

Here, in the right-hand side of the equation, we view σ as a word $\sigma(1) \dots \sigma(n)$ in the letters $1, \dots, n$ and embed $\mathbb{Z}S_n$ in the tensor algebra on the integers $T(\mathbb{N})$ (that is, we view the integers as the *letters* of an alphabet). In particular, $\hat{\delta}^T$ is the unshuffle coproduct (see section 4) and $\hat{\delta}^T(\sigma) = \sum_{S \sqcup T = [n]} \sigma|_S \otimes \sigma|_T$, where we write $\sigma|_S$ for $\sigma(s_1) \dots \sigma(s_{|S|})$.

Let us prove that, given S and T such that $S \sqcup T = [n]$, $|S| = k$, we have:

$$\bar{\delta}_{k,n-k} \circ \sigma_{S,T}(\sigma) = (\text{st} \circ \sigma|_S) \otimes (\text{st} \circ \sigma|_T).$$

The proof will follow, by definition of $\hat{\delta}$. We have:

$$\begin{aligned} \sigma \cdot \sigma_{S,T} &= \sigma \cdot (s_1, \dots, s_k, t_1, \dots, t_{n-k}) \\ &= (\sigma(s_1), \dots, \sigma(s_k), \sigma(t_1), \dots, \sigma(t_{n-k})) \end{aligned}$$

and therefore:

$$\bar{\delta}_{k,n-k}(\sigma \cdot \sigma_{S,T}) = \text{st}(\sigma|_S) \otimes \text{st}(\sigma|_T).$$

Proposition 19. *The cosymmetrized coproduct is cocommutative.*

The proposition follows immediately from the description of $\hat{\delta}$ above.

Theorem 20. *The bialgebra \hat{S} is a cocommutative connected bialgebra. It is therefore the enveloping algebra of its primitive part.*

Theorem 21. *The Lie algebra $\text{Prim}(\hat{S})$ is a free Lie algebra.*

The theorem follows from the observation that the algebra \hat{S} is free and from the lemma below. Its generators are the connected permutations, that is the permutations $\sigma \in S_n$ that can not be written as a product $\beta \times \gamma$, where β and γ are non trivial, see [12].

Lemma 22. *Let A be a graded connected cocommutative bialgebra over R such that its graded components are finite dimensional. If A is free as a graded algebra, then the Lie algebra of primitive elements in A is a free Lie algebra.*

We use in the proof the methods of [10]. Let $G = \{g_1, \dots, g_n, \dots\}$ be a free set of generators for A , ordered so that $\text{deg}(g_i) \leq \text{deg}(g_j)$ if $i \leq j$. We write e^1 for the logarithm of the identity of A in the convolution algebra of linear endomorphisms of A . Then, according to [10, Prop. III, 5], for any $g_i \in G$, $g_i = e^1(g_i) + g'_i$, where g'_i is a polynomial in the elements of A of degree strictly less than $\text{deg}(g_i)$. The set of the $e^1(g_i)$'s is then a set of generators of A . Besides, $e^1(g_i)$ is a primitive element in A (see [9], [10]) and is nonzero since g_i belongs to a set of free generators of A . An obvious induction argument using the ordering on G shows that the $e^1(g_i)$'s are algebraically independent (in the noncommutative sense), and form therefore a free set G' in A .

They generate a free Lie algebra $\text{Lie}(G')$ that is a Lie subalgebra of the Lie algebra $\text{Lie}(A)$ of primitive elements in A . Moreover, since G' generates A , the enveloping algebra of $\text{Lie}(G')$ can be identified with A , that is: $A = U(\text{Lie}(G'))$. Since, by the Cartier–Milnor–Moore theorem, $A = U(\text{Lie}(A))$ and since $\text{Lie}(G') \subset \text{Lie}(A)$, we have: $\text{Lie}(A) = \text{Lie}(G')$. This proves the lemma.

Recall that, according to Proposition 11, the set of primitive elements of the twisted bialgebra S embeds into the set of primitive elements of \hat{S} . Therefore, the images under $\sigma \rightarrow \sigma^{-1}$ of Lie elements are primitive elements of \hat{S} according to Proposition 17. Hence the space of primitive homogeneous elements of degree n in \hat{S} is at least of degree $(n - 1)!$. It is actually much bigger. Indeed, if we denote α_n its dimension, then the theorem of Poincaré–Birkhoff–Witt implies that

$\prod_{n \geq 1} (1 - \alpha_n x^n)^{-1} = \sum_{n \geq 0} n! x^n$. This identity allows us to compute the first few values of the α_n , for $n = 1, \dots, 11$: 1, 1, 4, 17, 92, 572, 4156, 34159, 314368, 3199844, 35703996. It seems that α_n is much closer to $n!$ than to $(n-1)!$. This is indeed true, and we show that α_n is asymptotically equivalent to $n!$. More precisely, we have

$$0 < 1 - \frac{\alpha_n}{n!} < \frac{1}{n} + \frac{8}{(n-2)^2}.$$

Indeed, it is known that the sum of the inverses of the $n+1$ binomial coefficients $\binom{n}{k}$ tends to 2 when n tends to infinity. We have actually

$$\begin{aligned} \sum_{1 \leq k \leq \frac{n}{2}} \binom{n}{k}^{-1} &\leq n^{-1} + \binom{n}{2}^{-1} + n \binom{n}{3}^{-1} \\ &= n^{-1} + \frac{2}{n(n-1)} + n \frac{6}{n(n-1)(n-2)} < n^{-1} + 8(n-2)^{-2}. \end{aligned}$$

Now, according to the Poincaré-Birkhoff-Witt theorem, $n! - \alpha_n$ is less than or equal to the sum of all $p!q!$, $p+q=n$, $1 \leq p \leq q$. If we divide by $n!$, we obtain the sum of inverses of binomial coefficients considered above. Hence $n! - \alpha_n < n!(n^{-1} + 8(n-2)^{-2})$ and $\alpha_n > n!(1 - n^{-1} - 8(n-2)^{-2})$, which implies the displayed inequality and shows that α_n is asymptotically equal to $n!$.

REFERENCES

- [1] M. G. Barratt, *Twisted Lie algebras*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 9–15. MR 80d:55017
- [2] L. Geissinger, *Hopf algebras of symmetric functions and class functions*, Combinatoire et représentation du groupe symétrique (Actes Table Ronde C.N.R.S., Univ. Louis-Pasteur Strasbourg, Strasbourg, 1976), Lecture Notes in Math., vol. 579, Springer, Berlin, 1977, pp. 168–181. MR 58 #22149
- [3] I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. Math. **112** (1995), no. 2, 218–348. MR 96e:05175
- [4] A. Joyal, *Foncteurs analytiques et espèces de structures*, Combinatoire énumérative (Montreal, Que., 1985/Quebec, Que., 1985), Lecture Notes in Math., vol. 1234, Springer, Berlin, 1986, pp. 126–159. MR 89b:05014
- [5] D. Krob, B. Leclerc, and J.-Y. Thibon, *Noncommutative symmetric functions. II. Transformations of alphabets*, Internat. J. Algebra Comput. **7** (1997), no. 2, 181–264. MR 98f:05148
- [6] S. Mac Lane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 2001j:18001
- [7] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra **177** (1995), no. 3, 967–982. MR 97d:05277
- [8] J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. (2) **81** (1965), 211–264. MR 30 #4259
- [9] F. Patras, *La décomposition en poids des algèbres de Hopf*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 4, 1067–1087. MR 95d:16054
- [10] F. Patras, *L'algèbre des descentes d'une bigèbre graduée*, J. Algebra **170** (1994), no. 2, 547–566. MR 96a:16043
- [11] F. Patras and C. Reutenauer, *Lie representations and an algebra containing Solomon's*, J. Algebraic Combin. **16** (2002), no. 3, 301–314 (2003). MR 2003m:20012
- [12] S. Poirier and C. Reutenauer, *Algèbres de Hopf de tableaux*, Ann. Sci. Math. Québec **19** (1995), no. 1, 79–90. MR 96g:05146

- [13] C. Reutenauer, *Free Lie algebras*, London Mathematical Society Monographs. New Series, vol. 7, The Clarendon Press Oxford University Press, New York, 1993. MR 94j:17002
- [14] C. R. Stover, *The equivalence of certain categories of twisted Lie and Hopf algebras over a commutative ring*, J. Pure Appl. Algebra 86 (1993), no. 3, 289–326. MR 94e:16031

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