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Corresponding Author	Family Name	**Reutenauer**
	Particle	
	Given Name	**Christophe**
	Suffix	
	Division	Département de mathématiques
	Organization	Université du Québec à Montréal
	Address	Montréal, QC, Canada
	Email	reutenauer.christophe@uqam.ca
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Combinatorics on Words and the Theory of Markoff

Christophe Reutenauer

Abstract This is a survey on the theory of Markoff, in its two aspects: quadratic forms (the original point of view of Markoff), approximation of reals. A link wih combinatorics on words is shown, through the notion of Christoffel words and special palindromes, called central words. Markoff triples may be characterized, by using some linear representation of the free monoid, restricted to these words, and Fricke relations. A double iterated palindromization allows to construct all Markoff numbers and to reformulate the Markoff numbers injectivity conjecture (Frobenius, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin 26:458–487, 1913).

1 Introduction

In a short article written in Latin in 1875, Christoffel [11] introduced a family of words on a two letter alphabet, that we call *Christoffel words*. Shortly after, they were also considered by Smith [36], who did not know, as he says and regrets, Christoffel's work. These words were followed in the twentieth century by the theory of Sturmian sequences, introduced in 1940 by Morse and Hedlund [28] in Symbolic Dynamics. More recently, there has been a lot of work, beginning by Jean Berstel and Aldo de Luca, on these words, from the point of view of Combinatorics on Words and also in Discrete Geometry, see among others [2, 7, 23].

Independently from Christoffel, Markoff (= Markov, famous for the Markov processes, but writing his name in the French way) wrote as young student two brilliant articles [26, 27] in 1879 and 1880, on the theory called now *Theory of Markoff*. This theory has two sides: it characterizes on one hand certain quadratic forms, and on the other, certain real numbers, defined by some extremal conditions (by their minima for quadratic forms, and by their rational approximations for

C. Reutenauer (✉)
Département de mathématiques, Université du Québec à Montréal, Montréal, QC, Canada
e-mail: reutenauer.christophe@uqam.ca

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real numbers). They are constructed using some special integers called *Markoff numbers*, which are among others characterized by a special Diophantine equation, the *Markoff equation*.

Looking at the subsequent literature, it is seen that Markoff's theory has visibly fascinated many mathematicians, which have developed, and often reproved one or the other side of the theory: Hurwitz [21], Frobenius [19], Perron [30], Remak [31], Dickson [17], Cassels [9, 10], Cohn [12], Bombieri [5], and the list is much longer. Three books must be cited here: the unavoidable book by Cusick and Flahive [14], the book by Perrine [29] who gives among others a lot of matrix constructions, and the recent book by Aigner [1], celebrating the 100th anniversary of Frobenius' *injectivity conjecture for Markoff numbers*.

Markoff constructs the special quadratic forms that appear in his theory by using special patterns of 1s and 2s (in the continued fraction expansion); these patterns happen to be Christoffel words. The link between Christoffel words and the theory of Markoff was explicitly noted by Frobenius in 1913 [19], and somewhat forgotten until recently (but it was known to Caroline Series [35]). The scope of the present survey is to present Markoff's theory, from the point of view of Combinatorics on words, especially Christoffel words. In the two final sections, we review some tree constructions, and relate calculations made by Frobenius to the standard factorization of Christoffel words (this seems to be new).

The interested reader will find many results and proofs in the forthcoming book [33]. For the theory of Christoffel and related words, see also Chapter 2 of [25] and the first part of the book [3].

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2 Christoffel Words

A *lattice path* is a sequence of consecutive elementary steps in the plane; each *elementary step* is a segment $[(x, y), (x+1, y)]$ or $[(x, y), (x, y+1)]$, with $x, y \in \mathbf{Z}$.

Let p, q be relatively prime natural integers. Consider the segment from some integral point A to $B = A + (p, q)$ and the lattice path from A to B located below this segment and such that the polygon delimited by the segment and the path has no interior integer point.

Given a totally ordered alphabet $\{a < b\}$, the *lower Christoffel word of slope* q/p is the word in the free monoid[1] $\{a, b\}^*$ coding the above path, where a (resp. b) codes an horizontal (resp. vertical) elementary step. See Fig. 1, where is represented the path with $(p, q) = (7, 4)$ corresponding to the Christoffel word *aabaabaabab* of slope 4/7. Note that the slope of a Christoffel word is equal to the slope of the

[1] The free monoid A^* is the set of words (= strings = finite sequences) on the set A, including the empty one; this is a monoid, the product of two words being the concatenation.

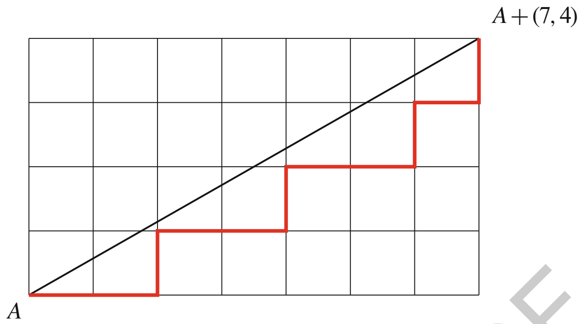


Fig. 1 The lower Christoffel words *aabaabaabab* of slope 4/7

segment in the plane delimited by the extreme points of the corresponding discrete path.

We say that the above path, and the lower Christoffel word, *discretizes from below* the segment AB.

The upper Christoffel word of slope q/p is defined similarly, by considering the lattice path located above the segment. Since the rectangle with opposite vertices A and B and sides parallel to the coordinate lines has a symmetry around its center, it follows that *the upper Christoffel word of a given slope is the reversal* $\tilde{w}$ *of the lower Christoffel word* w *of the same slope*. It is known also that *a lower Christoffel word* w *and the corresponding upper Christoffel word* $\tilde{w}$ *are conjugate*[2] *in the free monoid* $\{a < b\}^*$. See [3] Lemma 2.7.

Clearly, the number of a's in the lower and upper Christoffel word of slope q/p is p, while the number of b's is q. In particular, $|w|_a, |w|_b$ are relatively prime when w is a lower or upper Christoffel word and w cannot be a nontrivial power of another word.

The letters a and b are Christoffel words. The other Christoffel words are called *proper*. The words $a^n b$ and ab^n, for $n \geq 0$, are lower Christoffel words.

On the path defining the Christoffel word, consider the integral point, not equal to the first nor to the last, which is the closest to the diagonal AB of the rectangle. This defines a factorization of the Christoffel word, called its *standard factorization*. In the example of Fig. 1, the point is $A+(2, 1)$, and the factorization is $aab.aabaabab$. It follows from a theorem of Borel and Laubie that the two factors are themselves Christoffel words, and that this factorization is unique [3, 6] Theorem 3.3.

[2]This means that $w = uv$ and $\tilde{w} = vu$ for some words u, v.

3 Markoff Triples and Numbers

A *Markoff triple* is a multiset $\{x, y, z\}$ of positive integers satisfying the *Markoff equation*

$$x^2 + y^2 + z^2 = 3xyz.$$

Examples are $\{1, 1, 1\}$, $\{1, 1, 2\}$ and $\{1, 2, 5\}$. We sometimes use ordered triples for representing Markoff triples. A Markoff triple is called *proper* if the three numbers are distinct. Otherwise we call it *improper*. The improper Markoff triples are $\{1, 1, 1\}$ and $\{1, 1, 2\}$. A *Markoff number* is an element of a Markoff triple.

Consider the monoid homomorphism μ from the free monoid $\{a, b\}^*$ into $SL_2(\mathbb{Z})$ defined by

$$\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$

The matrix construction of the Markoff triples as shown in the following theorem was obtained by Cohn [12, 13]. He used the third Fricke relation (see the lemma below), having noted the striking analogy between Markoff's equation and this relation. Uniqueness was noticed by Bombieri [5] Theorem 26, and independently by the author [32] Theorem 1.

Theorem 3.1 *The mapping sending each Christoffel word w with standard factorization uv onto the multiset $\{\frac{1}{3}Tr(\mu(u)), \frac{1}{3}Tr(\mu(v)), \frac{1}{3}Tr(\mu(w))\}$ is a bijection from the set of proper lower Christoffel words onto the set of proper Markoff triples.*

It is useful to know that for w a lower Christoffel word, one has $\frac{1}{3}Tr(\mu(w)) = \mu(w)_{12}$ (see [3] Lemma 8.7). The Fricke relations are given in the following lemma.

Lemma 3.1 (Fricke relations [18], (6) p. 91) *Let A, B be matrices in $SL_2(\mathbb{Z})$. Then:* $\mathrm{Tr}(A^2B) + \mathrm{Tr}(B) = \mathrm{Tr}(A)\,\mathrm{Tr}(AB)$, $\mathrm{Tr}(AB^2) + \mathrm{Tr}(A) = \mathrm{Tr}(AB)\,\mathrm{Tr}(B)$ *and* $\mathrm{Tr}(A)^2 + \mathrm{Tr}(B)^2 + \mathrm{Tr}(AB)^2 = \mathrm{Tr}(A)\,\mathrm{Tr}(B)\,\mathrm{Tr}(AB) + \mathrm{Tr}(ABA^{-1}B^{-1}) + 2$.

It follows from Theorem 3.1 that for each Markoff number m, there exists a lower Christoffel word w such that $m = \frac{1}{3}Tr(\mu(w)) = \mu(w)_{12}$. We say that m is the Markoff number *associated to the Christoffel word w*. The so-called *conjecture of Frobenius* [19], also called *Markoff numbers injectivity conjecture*, is the following open question[3]: is the mapping $w \mapsto m$ injective? Of course, the theorem has a striking analogy with this conjecture, since the former asserts that the mapping which to w associates its Markoff triple is bijective. Some partial answers to the

[3] Frobenius states it, in two different forms, as an open problem, not a conjecture [19] p. 601 and 614.

conjecture have been given (see the book [1] by Martin Aigner), but the general case seems to be very difficult.

The Frobenius conjecture is equivalent to the conjecture that for each Markoff number m, there is a unique Markoff triple of which m is the maximum, see [1] p. 39 (this works also for the improper triples). For example, 1, 2, 5 are respectively the unique maxima of (1, 1, 1), 1, 1, 2), (1, 2, 5).

4 Lagrange Number of a Real Number

Let x be an irrational real number. Consider the set of real numbers L such that the inequality $|x - p/q| < 1/Lq^2$ holds for infinitely many rational numbers p/q.

Define $L(x)$ to be the supremum of all these L. It is called the *Lagrange number* of x.

Let x be represented by its infinite continued fraction $[a_0, a_1, a_2, \ldots]$, and define $x_n = [a_n, a_{n+1}, a_{n+2}, \ldots]$; moreover, for $n \geq 1$, define $y_n = [a_n, \ldots, a_1]$. Finally, for $n \geq 2$, let $\lambda_n(x) = x_n + y_{n-1}^{-1}$. We have

$$\begin{aligned}\lambda_n(x) &= [a_n, a_{n+1}, \ldots] + [a_{n-1}, \ldots, a_1]^{-1}\\ &= a_n + [a_{n+1}, a_{n+2}, \ldots]^{-1} + [a_{n-1}, \ldots, a_1]^{-1}\\ &= [a_{n+1}, a_{n+2}, \ldots]^{-1} + [a_n, a_{n-1}, \ldots, a_1]. \qquad (1)\end{aligned}$$

For $n = 1$, we define by the last equation: $\lambda_1(x) = [a_2, a_3, \ldots]^{-1} + a_1 = x_1$. The next result is essential for the determination of the Lagrange number of a sequence; it is stated without proof by Hurwitz [21] p. 283, see [1] p. 23 for a proof.

Theorem 4.1

$$L(x) = \limsup_{n\to\infty} \lambda_n(x).$$

The main tool in the proof is the following classical identity (where p_n/q_n is the n-th convergent of x)

$$|x - \frac{p_n}{q_n}| = \frac{1}{\lambda_{n+1}(x) q_n^2}. \qquad (2)$$

Recall that two irrational real numbers are called *equivalent* if their expansions into continued fractions coincide after some rank (which may be not the same rank for both numbers). It follows from the previous theorem that in this case they have the same Lagrange number.

5 Main Technical Result

Motivated by the previous result, we define for any infinite word $s = a_0a_1a_2\cdots$ over the $\mathbb{P}$ set of positive natural integers

$$\lambda_i(s) = [a_{n-1}, a_{n-2}, \ldots, a_1]^{-1} + [a_n, a_{n+1}, a_{n+2}, \ldots].$$

The next result has a strong similarity with previous results involving doubly infinite words, as used by Markoff and subsequent authors (for example Dickson and Bombieri). It was however tempting to find a version for infinite words, since continued fractions are such words. This result will be used to prove Markoff's theorems for continued fractions and Markoff's theorem on quadratic forms. Denote by χ the monoid homomorphism from the free monoid $\{a, b\}^*$ into the free monoid $\{1, 2\}^*$ sending a onto 11 and b onto 22. For a nonempty word $w = a_0 \ldots a_{n-1}$, we denote by w^∞ the infinite word $b_0b_1b_2b_3\cdots$ whose letter b_i in position i satisfies $b_i = a_{i \bmod n}$.

Theorem 5.1 *Let s be an infinite word over $\mathbb{P}$ such that for some I and some $\theta < 3$ one has $\lambda_i(s) < \theta$ for any $i \geq I$. Then $s = u\chi(w)^\infty$ for some lower Christoffel word w, and some word u whose length is bounded by a function depending only on I and θ.*

This result (except the bounds) may be deduced from similar results for bi-infinite words (see for example [1, 5]). A direct proof will be found in the forthcoming book [33].

6 Markoff's Theorem for Approximations

For the results in these sections, see among others [10] Theorem III p.41, [5] Theorem 1 and [1] Theorem p.185 (and also [33]).

If $s = b_0b_1b_2b_3\cdots$ is an infinite word on $\mathbb{P}$, we denote by $[s]$ the real number whose expansion into continued fractions is $[b_0, b_1, b_2, b_3, \ldots]$.

Theorem 6.1 *Let x be an irrational real number. Then its Lagrange number $L(x)$ is < 3 if and only if x is equivalent to some number $x_w = [\chi(\tilde{w})^\infty]$ (or equivalently to $[\chi(w)^\infty]$) for some lower Christoffel word w. In this case, let m be the Markoff number $\mu(w)_{12} = \frac{1}{3}Tr(\mu(w))$ associated to w. Then $L(x) = L(x_w) = \sqrt{9 - \frac{4}{m^2}}$ and $x_w = \frac{p-s}{2m} + \frac{1}{2}\sqrt{9 - \frac{4}{m^2}}$ with $\mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and therefore $m = q = \frac{1}{3}(p+s)$.*

Note that for some technical reasons, we have chosen x_w as in the statement. One could choose $[\chi(w)^\infty]$ instead, at the cost of transposing $\mu(w)$, or taking upper Christoffel words instead of lower ones. This is not fundamental.

Markoff's theorem is often stated as a series of progressively better approximations, with exceptions, the first exception being the golden ratio and the numbers equivalent to it. Formally, this is the following result. Informally, see the examples following it.

Corollary 6.1 *Let M be a finite set of Markoff numbers, such that for any Markoff numbers n, m, if $n < m$ and $m \in M$, then $n \in M$. Let W be the finite set of all lower Christoffel words corresponding to the Markoff numbers in M (in other words, $W = \{w \in W | \mu(w)_{12} \in M\}$). Let m be the smallest Markoff number not in M. Then for each irrational real number not equivalent to any x_w, $w \in W$, there are infinitely many rational approximations p/q of x such that $|x - p/q| < 1/\sqrt{9 - \frac{4}{m^2}}q^2$.*

We give three examples. First, let $M = \emptyset$. Then $m = 1$, $W = \emptyset$ and $\sqrt{9 - \frac{4}{m^2}} = \sqrt{5}$. We obtain that each real irrational number has infinitely many rational approximations satisfying $|x - p/q| < 1/\sqrt{5}q^2$. This is a theorem of Hurwitz [21], Satz 1 p. 279, that we state as corollary.

Corollary 6.2 *For each real number x there are infinitely many rational fractions $\frac{p}{q}$ such that $|x - p/q| < 1/\sqrt{5}q^2$.*

Now let $M = \{1\}$. Then $m = 2$, $W = \{a\}$, $\sqrt{9 - \frac{4}{m^2}} = \sqrt{8}$. Moreover $x_a = \frac{\sqrt{5}+1}{2}$, the *golden ratio*. We obtain that each real irrational number x not equivalent to x_a has infinitely many rational approximations p/q satisfying $|x - p/q| < 1/\sqrt{8}q^2$.

Finally, let $M = \{1, 2\}$. Then $W = \{a, b\}$, $m = 5$, $\sqrt{9 - \frac{4}{m^2}} = \frac{\sqrt{221}}{5}$, $x_b = 1 + \sqrt{2}$. We obtain that each irrational real number not equivalent to x_a nor to x_b has infinitely many rational approximations satisfying $|x - p/q| < 1/\frac{\sqrt{221}}{5}q^2$.

7 Markoff's Theorem for Quadratic Forms

For the results in these sections, see among others [10] Theorem II p.39, [17] Theorem 62 p.79 and seq., [14] theorem 6 p.10 (and also [33]).

A *real binary quadratic form* is a polynomial $f(x, y) = \alpha x^2 + \beta xy + \gamma y^2$ in the variables x, y and real coefficients α, β, γ not all zero. Its *discriminant* is $d(f) = \beta^2 - 4\alpha\gamma$. If the latter number is positive, the form is called *indefinite*.

We are interested here in the *greatest lower bound* of such a form, defined by $L(f) = \inf\{|f(x, y|, x, y \in \mathbb{Z}, (x, y) \neq (0, 0)\}$. We say that the lower bound is *attained* if there exist $(x, y) \in \mathbb{Z}^2 \setminus (0, 0)$ such that $L(f) = f(x, y)$. If the coefficients of f are integers (the interesting case), then the bound is clearly attained.

Two quadratic forms are *equivalent* if each of them is obtained from the other by a change of variables over $\mathbb{Z}$.

Let w be a lower Christoffel word and let $\mu(w) = \begin{pmatrix} p & m \\ r & s \end{pmatrix}$, where m is the Markoff number associated to w and therefore $m = \frac{1}{3}(p+s)$. Define the associated *Markoff quadratic form* by $f_w(x, y) = mx^2 + (s-p)xy - ry^2$.

Theorem 7.1 *Let $f(x, y)$ be a indefinite binary quadratic form. Assume that its greatest lower bound $L(f)$ and its discriminant $d(f)$ satisfy the inequality $\sqrt{d(f)} < 3L(f)$. Then f is equivalent to a multiple of some Markoff form f_w. Let m be the Markoff number associated to the lower Christoffel word w. One has $d(f_w) = 9 - 4m^2$, $L(f_w) = f_w(1, 0) = m$, so that the lower bounds of f_w and f are attained, and $\frac{\sqrt{d(f)}}{L(f)} = \frac{\sqrt{d(f_w)}}{L(f_w)} = \sqrt{9 - \frac{4}{m^2}}$.*

Remark 7.1 The first inequality in the theorem implies that $L(f)$ does not vanish (that is, there is no pair $(x, y) \neq (0, 0)$ of integers such that $f(x, y) = 0$). If we consider only quadratic forms satisfying this, then we see below (Corollary 7.2) that $\sqrt{d(f]}/L(f)$ is always at least equal to $\sqrt{5}$ (which is smaller that 3). Markoff's theorem is about such quadratic forms satisfying $\sqrt{d(f]}/L(f) < 3$. Note that there exist quadratic forms with $\sqrt{d(f]}/L(f)$ arbitrarily large: this set of real numbers, for all possible f, is called the *Markoff spectrum*, see [14].

This theorem is also stated as a series of progressively better inequalities, with exceptions, as follows.

Corollary 7.1 *Let M be a finite set of Markoff numbers, such that for any Markoff numbers n, m, if $n < m$ and $m \in M$, then $n \in M$. Let W be the finite set of all lower Christoffel words corresponding to the Markoff numbers in M (in other words, $W = \{w \in W | \mu(w)_{12} \in M\}$). Let m be the smallest Markoff number not in M. The for each indefinite binary quadratic form $f(x, y)$, not equivalent to a multiple of any form f_w, $w \in W$, one has $\sqrt{d(f)} \geq \sqrt{9 - \frac{4}{m^2}} L(f)$.*

We give several examples. First, let $M = \emptyset$. Then $m = 1$, $W = \emptyset$ and $\sqrt{9 - \frac{4}{m^2}} = \sqrt{5}$. We obtain a result due to Korkine and Zolotareff [24].

Corollary 7.2 *Let $f(x, y)$ be a indefinite binary quadratic form. Then $\sqrt{d(f)} \geq \sqrt{5} L(f)$.*

Now let $M = \{1\}$. Then $m = 2$, $W = \{a\}$, $\sqrt{9 - \frac{4}{m^2}} = \sqrt{8}$. We have $\mu(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ so that $f_a = x^2 - xy - y^2$. Thus we obtain the following result, also due to Korkine and Zolotareff [24].

Corollary 7.3 *Let $f(x, y)$ be a indefinite binary quadratic form. If f is not equivalent to a multiple of the form $f_a = x^2 - xy - y^2$, then $\sqrt{d(f)} \geq \sqrt{8} L(f)$.*

These two results are mentioned by Markoff [26], who gives them as the motivation of his own work. The next example is $M = \{1, 2\}$. Then $W = \{a, b\}$, $m = 5$, $\sqrt{9 - \frac{4}{m^2}} = \frac{\sqrt{221}}{5}$, $x_b = 1 + \sqrt{2}$. Since $\mu(b) = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$, we have $f_b = 2x^2 - 4xy - 2y^2$. Thus, for each form not equivalent to a multiple of $f_a = x^2 - xy - y^2$ nor of $f_b = 2x^2 - 4xy - 2y^2$, one has $\sqrt{d(f)} \geq \frac{\sqrt{221}}{5} L(f)$.

8 Several Binary Complete Infinite Trees

The set of Markoff triples may be organized as the set of nodes of an infinite binary tree; this follows from an operation on triples, already present in Markoff's work, that transforms each triple in three other ones: in the tree this corresponds to going to its parent, or to one of its two children. Other trees appear in the literature. All these trees are specialization of the first one, which we define now.

The nodes of the first tree are the pairs (u, v) where uv is a lower Christoffel word with its standard factorization. It was introduced by Jean Berstel and Aldo de Luca in [2]. Its root is (a, b) and the tree is constructed using the rule given in Fig. 2.

We call this tree the *tree of Christoffel pairs*, see Fig. 3.

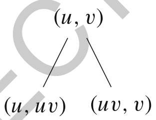


Fig. 2 The rule for constructing the tree of Christoffel pairs

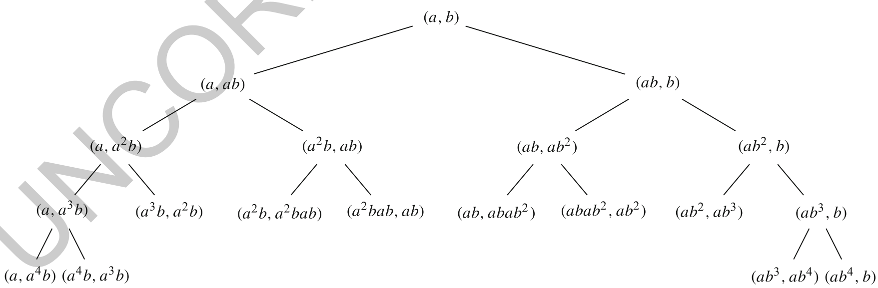


Fig. 3 The tree of Christoffel pairs

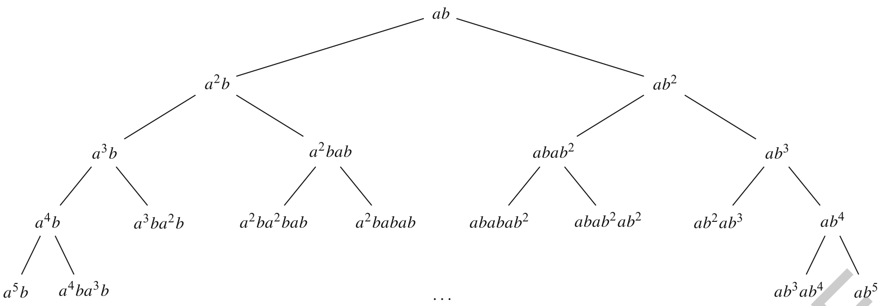


Fig. 4 The tree of Christoffel words

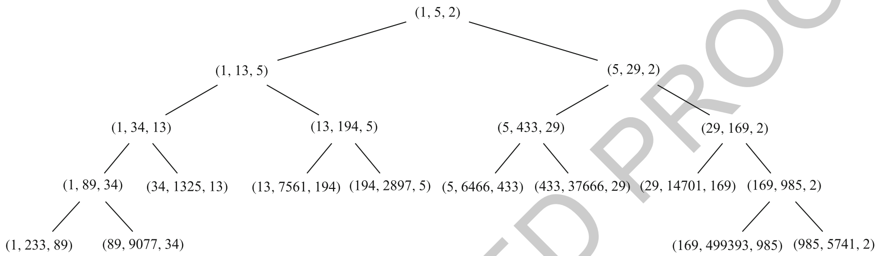


Fig. 5 The tree of Markoff triples

The second tree, the *tree of (lower) Christoffel words* is obtained from the previous one by replacing each node (u, v) by uv. Its nodes are exactly all lower Christoffel words. See Fig. 4.

This tree may constructed directly by taking as root ab, and any other node w is obtained as follows: consider the path from w towards the root.

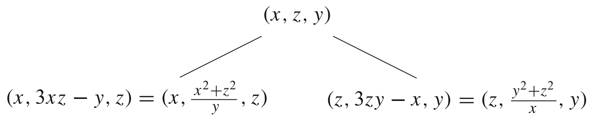
(i) Suppose that w is not on the two extreme branches of the tree; then this path has north-west steps and north-east steps; let u be the node after the first north-west step and v be the node after the first north-east step. Then $w = uv$.

(ii) If w is on the left (respectively right) extreme branch, then no north-west (respectively north-east) step exists. Choose $u = a$ and v as in (i) (respectively u as in (i) and $v = b$).

For example, the node $w = a^2babab$ is the product of the nodes $u = a^2bab$ and $v = ab$.

The third tree is called the *tree of Markoff triples*. It is obtained by replacing each node (u, v) in the tree of Christoffel pairs by the triple $(\mu(u)_{12}, \mu(uv)_{12}, \mu(v)_{12}) = (\frac{1}{3}Tr(\mu(u)), \frac{1}{3}Tr(\mu(uv)), \frac{1}{3}Tr(\mu(v)))$. By Theorem 3.1, the nodes of this tree are the proper Markoff triples, each of which is represented by some ordered triple. See Fig. 5.

Fig. 6 The rule for building the tree of Markoff triples



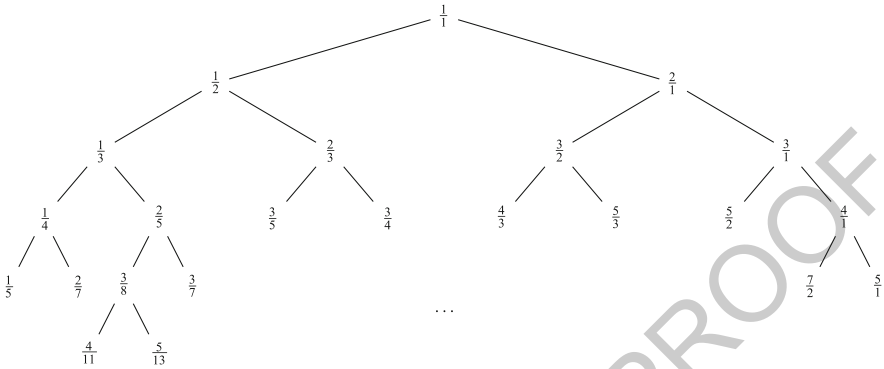


Fig. 7 The Stern-Brocot tree

This tree may be constructed directly by the rule given in Fig. 6, as follows from the Fricke relations, see Lemma 3.1.

The fourth tree, the *Stern-Brocot tree*, is obtained, following [2], from the tree of Christoffel pairs by replacing each node (u, v) by the *slope* of the word uv, that is the quotient of its number of b's divided by its number of a's. The nodes of the Stern-Brocot tree are the positive rational numbers. See Fig. 7.

The Stern-Brocot tree may be constructed directly, by mimicking the direct construction of the tree of Christoffel pairs: its root is $\frac{1}{1}$; consider the path from some other node $\frac{q}{p}$ to the root, and let $\frac{q'}{p'}$ (resp. $\frac{q''}{p''}$) be the node immediately after the first north-west (resp. north-east) step in this path (if no such step exists, then take $\frac{0}{1}$ resp. $\frac{1}{0}$). Then $\frac{q}{p}$ is the *mediant* of $\frac{q'}{p'}$ and $\frac{q''}{p''}$, that is $p = p' + p''$ and $q = q' + q''$. For example, $\frac{3}{4}$ is the mediant of $\frac{2}{3}$ and $\frac{1}{1}$, and $\frac{1}{3}$ is the mediant of $\frac{0}{1}$ and $\frac{1}{2}$.

The Stern-Brocot tree is a variant of the notion of continued fractions. It contains all *semi-convergents* of any real number. See [20].

The fifth tree is the *Raney tree* of [2] (see also [8]). It is obtained from the tree of Christoffel pairs by replacing each node (u, v) by the rational number $\frac{|u|}{|v|}$. The nodes of the Raney tree are the positive rational numbers (Fig. 8).

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This tree may be constructed directly by applying the following rule: the root is $\frac{1}{1}$; if $\frac{q}{p}$ is a node, then its left child is $\frac{q}{p+q}$ and its right child is $\frac{p+q}{p}$. This follows directly from the rule for constructing the tree of Christoffel pairs, see Fig. 2.

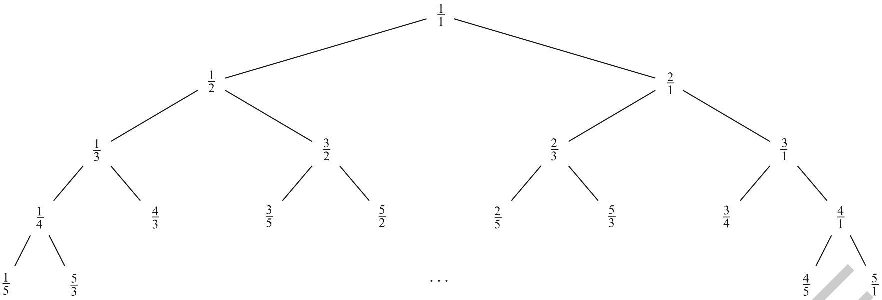

Fig. 8 The Raney tree

9 Frobenius Congruences Through Christoffel Words

In [19] p.602, Frobenius establishes several congruences related to a given Markoff triple, see also [1] p.33. We show that these congruences may be obtained through the standard factorization of Christoffel words.

Consider the monoid homomorphism ω from the free monoid $\{a, b\}^*$ into $SL_2(\mathbb{Z})$ defined by

$$\omega(a) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ \omega(b) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

This homomorphism is related to the homomorphism μ of Sect. 3 as follows.

Lemma 9.1 *Let* $P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. *Then for any word* $x \in \{a, b\}^*$, *one has* $\mu(x) = P^{-1}(\omega \circ G(x))P$ *where* G *is the endomorphism of the free monoid* $\{a, b\}^*$ *sending* a *onto* a *and* b *onto* ab. *Moreover* $P^{-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 & 1 \end{pmatrix} P = \begin{pmatrix} 2 & 1 \end{pmatrix}$.

We borrowed the notation G (meaning "gauche") from the notations for Sturmian morphisms as presented by Berstel and Séébold in [25].

Proof Clearly, $\begin{pmatrix} 1 & 1 \end{pmatrix} P = \begin{pmatrix} 2 & 1 \end{pmatrix}$. One has $P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. Hence $P^{-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Since $x \mapsto P^{-1}(\omega \circ G(x))P$ and μ are homomorphisms, it suffices to show that they coincide for $x = a$ and $x = b$. One has

$$P^{-1}\omega(G(a))P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \mu(a)$$

and

$$P^{-1}\omega(G(b))P = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} = \mu(b).$$

□

It is easy to see that if $w = amb$ is a lower Christoffel word, then m is a palindrome. These palindromes are called *central words*. We use below the *iterated palindromization mapping*, denoted Pal: it is a bijection from $\{a, b\}^*$ onto the set of central words. One defines first the *palindromic closure* $u^{(+)}$ of a word: it is the shortest palindrome having u as prefix; it is determined by the equalities $u = ps$, $u^{(+)} = ps\tilde{p}$, where s is the longest palindromic suffix of u and where $\tilde{p}$ is the *reversal* of p. Then Pal is defined recursively by $Pal(1) = 1$ (the empty word) and $Pal(ux) = (Pal(u)x)^{(+)}$ for any word u and any letter x. All these notions and results are due to Aldo de Luca [15].

For example, for the lower Christoffel word $aabaabaabab$, the associated central word is the palindrome $abaabaaba$, which is equal to $Pal(abaa)$: indeed, $Pal(a) = a^{(+)} = a$, $Pal(ab) = (a\underline{b})^+ = aba$, $Pal(aba) = (ab\underline{aa})^+ = abaaba$ and $Pal(abaa) = (ab\underline{aabaa})^+ = abaabaaba$, where the longest palindromic suffixes have been underlined.

The next result, which characterizes Markoff numbers, is due to Laurent Vuillon and the author [34].

Corollary 9.1 *Let v be any word in $\{a, b\}^*$ and w be the Christoffel word $w = aPal(v)b$. Then the Markoff number $\mu(w)_{12}$ is equal to the length of the lower Christoffel word $a(Pal \circ \theta \circ Pal(av))b$.*

Here θ denote the endomorphism of the free monoid that sends a onto ab and b onto ba, called the *Thue-Morse substitution*. Note that the mapping $\theta \circ Pal$ is the *iterated anti-palindromization mapping* (denoted $AntiPal$) of [16], Theorem 7.1: an anti-palindrome in $\{a, b\}^*$ is a word which is equal to the word obtained by exchanging a and b in its reversal; for example $abbaab$. One defines $AntiPal$

similarly to Pal, replacing in its definition palindromes by anti-palindromes. Hence, the previous result says that the mapping $v \mapsto 2+|Pal \circ AntiPal(v)|$ is a surjection form the free monoid onto the set of Markoff numbers (distinct form 1,2). The question of its injectivity is precisely the Frobenius conjecture (see [34]).

We use below the following result (see e.g. Corollary 3.2 in [4]).

let ν be the monoid homomorphism from the free monoid $\{a,b\}^*$ into the multiplicative monoid $SL_2(\mathbb{N})$ defined by

$$\nu(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \nu(b) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proposition 9.1 *Let $w = aPal(v)b$ be a lower Christoffel word and $w = w_1 w_2$ be its standard factorization. Let $\nu(v) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then the lengths of the words w_1, w_2 and $w_1 w_2$ are respectively $p+r$, $q+s$ and $p+q+r+s$.*

Proof of Corollary 9.1 In view of the previous proposition, it is enough to show that $\mu(aPal(v)b)_{12} = S\nu(\theta(Pal(av))$ where S sends each matrix onto the sum of its entries.

Note that $\nu\theta$ is equal to ω. Indeed,

$$\nu\theta(a) = \nu(ab) = \nu(a)\nu(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \omega(a).$$

Similarly $\nu\theta(b) = \omega(b)$.

Hence we have $S\nu\theta Pal(av) = S\omega Pal(av)$. Now we use the formula of Justin [22]: $Pal(av) = G(Pal(v))a$. Thus

$$S\omega Pal(av) = S(\omega G(Pal(v))\omega(a)) = \begin{pmatrix} 1 & 1 \end{pmatrix} \omega G(Pal(v)) \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that the product of the two last matrices is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Now, for any word x, we have

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \omega G(x) \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \end{pmatrix} PP^{-1}\omega G(x)PP^{-1} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

This is equal by Lemma 9.1 to

$$\begin{pmatrix} 2 & 1 \end{pmatrix} P^{-1}\omega G(x)P \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix} \mu(x) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Moreover,

$$\mu(aPal(v)b)_{12} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mu Pal(v) \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \end{pmatrix} \mu Pal(v) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

□

Corollary 9.2 *Consider a lower Christoffel word $w = w_1w_2$ with its standard factorization, and $(m, m_1, m_2) = (\mu(w)_{12}, \mu(w_1)_{12}, \mu(w_2)_{12})$ be the corresponding Markoff triple. Let $w = aPal(v)b$. Consider the Christoffel word $u = a(Pal \circ \theta \circ Pal(av))b$ and let its standard factorization be $u = u_1u_2$. Then the unique solution $x \in \{0, 1, \dots, m-1\}$ of the congruence $m_1x \equiv m_2 \mod m$ (resp. $m_2x \equiv m_1 \mod m$) is $x = |u_2|$ (resp $x = |u_1|$).*

Since the length of u is m, by Corollary 9.1, the sum of the two solutions of the congruences is m: modulo m they are opposite. Moreover, by squaring we obtain $m_i^2x^2 \equiv m_j^2 \mod m$, $\{i, j\} = \{1, 2\}$. By the Markoff equation we have $m_1^2 \equiv -m_2^2$, so that x is a square root of -1 modulo m.

An example is the following: $w = abb$, $w_1 = ab$, $w_2 = b$, $(m, m_1, m_2) = (29, 5, 2)$, $29^2 + 5^2 + 2^2 = 870 = 3 \cdot 29 \cdot 5 \cdot 2$, $v = b$, $Pal(ab) = aba$, $\theta Pal(ab) = abbaab$, $Pal\theta Pal(ab) = ababaababaababababaababaababa$, $u = aPal\theta Pal(ab)b = aababaababaababababaababaababab$, $u_1 = aababaababaababab$, $u_2 = aababaababab$, $|u_1| = 17$, $|u_2| = 12$, $m_1 \cdot |u_2| = 5 \cdot 12 = 60 = 2 \cdot 29 + 2 \equiv 2 = m_2 \mod 29$, $m_2 \cdot |u_1| = 2 \cdot 17 = 34 = 29 + 5 \equiv 5 = m_1 \mod 29$, $|u_1|^2 = 289 = -1 + 10 \cdot 29 \equiv -1 \mod 29$, $|u_2|^2 = 144 = -1 + 5 \cdot 29 \equiv -1 \mod 29$.

Proof Uniqueness follows from the fact that in a Markoff triple, the numbers are pairwise relatively prime ([1] Corollary 3.4). Let $\omega(Pal(av)) = \begin{pmatrix} h & i \\ j & k \end{pmatrix}$. It follows from the proposition and from the equality $\nu\theta = \omega$ that the lengths of u_1 and u_2 are respectively $h + j$ and $i + k$. From Lemma 9.1, we have $\omega(t) = P\mu G^{-1}(t)P^{-1}$ for any word t. By Justin's formula $G^{-1}(Pal(av)) = G^{-1}G(Pal(v))G^{-1}(a) = Pal(v)a$. Thus

$$\omega(Pal(av)) = P\mu(Pal(v))\mu(a)P^{-1} = P\mu(a)^{-1}\mu(aPal(v)b)\mu(b)^{-1}\mu(a)P^{-1}.$$

Since $\mu(a) = P^2$, we have $P\mu(a)^{-1} = P^{-1}$ and $\mu(b)^{-1}\mu(a)P^{-1} = \mu(b)^{-1}P = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$. Let $\mu(w) = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Thus

$$\omega(Pal(av)) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} r & s \\ p-r & q-s \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -r+3s & r-2s \\ -p+r+3q-s & p-r-2q+2s \end{pmatrix}.$$

It follows that the length of u_1 is $-p+3q$ and that of u_2 is $p-2q$. Now, let $\mu(w_1) = \begin{pmatrix} p_1 & q_1 \\ r_1 & s_1 \end{pmatrix}$ and $\mu(w_2) = \begin{pmatrix} p_2 & q_2 \\ r_2 & s_2 \end{pmatrix}$. We have $\mu(w) = \mu(w_1)\mu(w_2)$ and thus (since the determinants are equal to 1) $\mu(w_1) = \mu(w)\mu(w_2)^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} s_2 & -q_2 \\ -r_2 & p_2 \end{pmatrix}$, from which follows $q_1 = -pq_2 + qp_2$. Similarly $q_2 = -p_1q + q_1p$. Note that $m = q, m_1 = q_1, m_2 = q_2$. We deduce that $m_1|u_2| - m_2 = q_1(p-2m) - q_2 = p_1m - 2q_1m = m(p_1 - 2q_1)$. Likewise $m_2|u_1| - m_1 = q_2(-p+3m) - q_1 = -mp_2 + 3mq_2 = m(3q_2 - p_2)$. □

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