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M. Lothaire

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DIMENSIONS AND CHARACTERS OF THE DERIVED SERIES OF THE FREE LIE ALGEBRA

PAR

Christophe REUTENAUER

Abstract. — For each n and p, the symmetric group S_n acts on the homogeneous part of degree p of the $n-th$ term of the derived series of the free Lie algebra. A careful choice of a Hall basis together with the use of Joyal's theory of analytic functors allows to give formulas for the dimensions and the characters of these representations. They are obtained by plethystic iteration of the representations corresponding to the partitions $(1, p-1)$.

1. — Bases of the free Lie algebra

Let A be a set (called *alphabet*) and $\mathbf{Q}\langle A\rangle$ (resp. $\mathcal{L}(A)$) denote the free associative (resp. the free Lie) algebra generated by A over $\mathbf{Q}$; the algebra $\mathbf{Q}\langle A\rangle$ is also the algebra of noncommutative polynomials in the variables $a \in A$, or the tensor algebra

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},$$

where V is the vector space with basis A. It is known that $\mathcal{L}\langle A\rangle$ may be identified with the Lie sub-algebra of $\mathbf{Q}\langle A\rangle$ (with usual Lie bracket $[P,Q] = PQ - QP$) generated by the elements of A (see [2] or [4], cor. 5.3.9).

Let $M(A)$ denote the free magma over A, that is, the free nonassociative structure generated by A : each element of A may be viewed as a complete binary tree whose leaves are labelled in A; more formally, $M(A)$ is defined as the least set containing A, and containing (u, v) for each u, v in $M(A)$. Figure 1 shows an element of $M(a,b)$ in the two forms.

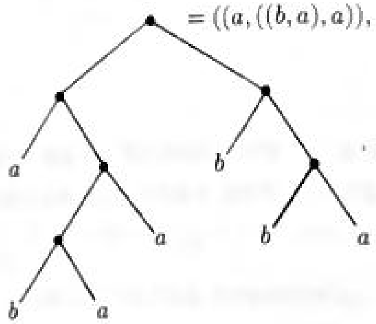

Fig. 1

Let $>$ be a total ordering of $M(A)$ satisfying

$$\forall u,v \in M(A),\ (u,v) > v. \tag{1}$$

$, u > v$

In other words, each tree in $M(A)$ is greater than its right subtree. One defines recursively the Hall set H relative to this order as the union of A with the set of trees of the form (u,v), with u,v in H and : either $u \in A$ or $u = (x,y)$ and $v \geqslant$ $\cancel{x} \cancel{\leq} y$. This Hall set H allows to define a basis of the free Lie algebra in the following way : let $\theta : M(A) \to \mathcal{L}(A)$ defined by $\theta(a) = a$ for any $a \in A$, and $\theta((u,v)) = [\theta(u), \theta(v)]$. In other words, interpret the (nonassociative) product of $M(A)$ as the Lie bracketing. Then $\theta(H)$ is a basis of $\mathcal{L}(A)$ (see [6], [4] ex. 5.4.3, [7] th. 1.2; these bases are more general than the Hall bases of [2]). Moreover, the elements $\theta(h)$, with $h \in H$, are all distinct. Note that $\theta(u)$ is an homogeneous polynomial of degree equal to the number of leaves of u, and that

$$\deg \theta((u,v)) = \deg \theta(u) + \deg \theta(v)$$

2. — Derived series

The *derived series* of $\mathcal{L}(A)$ is the decreasing sequence of Lie ideals of $\mathcal{L}(A)$ defined by

$$\mathcal{L}_0 = \mathcal{L}(A); \quad \mathcal{L}_{n+1} = [\mathcal{L}_n(A), \mathcal{L}_n(A)];$$

for any integer $n \geq 0$. We give a basis of each $\mathcal{L}_n$, by choosing a certain ordering of $M(A)$, and considering the associated Hall basis.

Define, for any tree in $M(A)$, its *level* (or Strahler number; see [8] for applications of this statistic on trees in combinatorics, computer science, hydrogeology and molecular biology) by the following rules :

$$\begin{aligned} \mathrm{l}(a) &= 0 && \text{if } a \in A; \\ \mathrm{l}((u,v)) &= \mathrm{l}(u) + 1 && \text{if } \mathrm{l}(u) = \mathrm{l}(v); \\ \mathrm{l}((u,v)) &= \sup(\mathrm{l}(u), \mathrm{l}(v)) && \text{if } \mathrm{l}(u) \neq \mathrm{l}(v). \end{aligned}$$

Note that the two last rules may be written as

$$\mathrm{l}((u,v)) = \sup(\mathrm{l}(u), \mathrm{l}(v)) + \delta(\mathrm{l}(u), \mathrm{l}(v)) \tag{2}$$

(Kronecker δ).

It is easy to verify that

$$\mathrm{l}(w) = n \implies \theta(w) \in \mathcal{L}_n$$

Note that $\mathrm{l}((u,v)) \geq \mathrm{l}(v)$. There exists a total ordering of $M(A)$ which is compatible with the level, that is,

$$\mathrm{l}(u) < \mathrm{l}(v) \implies u < v \tag{3}$$

and

$$(u,v) > v. \tag{4}$$

Indeed, first order separately each set $M_n = \{w \mid \mathrm{l}(w) = n\}$ in a way compatible with the number of leaves, and then extend these orders to $M(A)$ such that $u \in M_n$ and $v \in M_{n+1} \Rightarrow u < v$. Then condition (3) will be clearly satisfied, and for condition (4) : either $\mathrm{l}((u,v)) > \mathrm{l}(v)$, so (4) is satisfied or $\mathrm{l}(u,v) = \mathrm{l}(v)$, in which case (4) is also satisfied because (u,v) has more leaves than v.

Define now H_n to be the subset of H of trees of level $\geq n$.

THÉORÈME 1. — *The set $\theta(H_n)$ is a basis of $\mathcal{L}_n$.*

This theorem extends remark III. 1 of [6], and we use ideas of property III. 1 of the same article.

Proof. — It is enough to prove this result when A is finite. Moreover, we may identify h and $\theta(h)$, for $h \in H$, because $\theta|H$ is injective (see section 1).

We write $\mathrm{l}(P)$ for $\mathrm{l}(h)$ when $P = \theta(h)$.

1.— We show first that if P, Q in H are of level p, q, then $[P,Q]$ is a linear combination of T's in H with $\deg T = \deg P + \deg Q$ and $\mathrm{l}(T) \geq \sup(p,q) + \delta(p,q)$.

We use induction on the couple $(\deg P + \deg Q, \inf(P,Q))$, where these couples are ordered by : $(d',R') < (d,R)$ if either $d' < d$ or $(d' = d$ and

$R' > R$). As A is finite, there are only a finite number of (P', Q') with $\deg P' + \deg Q' = d$ and $\inf(P', Q') > R$: indeed, there are only a finite number of (Hall) trees with a prescribed number of leaves. Hence, the induction is correct.

We may suppose that $P > Q$. If $P \in A$ or $P = [R, S]$ with $S \leq Q$, then $[P, Q] \in H$ et we are done by definition of the level (*cf.* (2)).

We may thus suppose that $P = [R, S]$ with $R, S \in H$ and $R > S > Q$. This implies $p \neq q$: indeed if $p = q$, as $r \geq s \geq q$ (as $R > S > Q$; small letters indicate levels), one has by definition of the level :

$$p = r + \delta(r,s) \implies s \geq q = p = r + \delta(r,s) \geq s + \delta(r,s),$$

hence we have equality everywhere, that is, $r = s$ and $s \geq s + 1$, which is a contradiction.

Now, we have $p > q$ and $r \geq s \geq q$. By Jacobi's identity

$$[P, Q] = [[R, S], Q] = [[R, Q], S] + [R, [S, Q]].$$

By the induction hypothesis

$$[R, Q] = \sum \alpha_i A_i\,; \quad [S, Q] = \sum \beta_j B_j\,;$$

where $A_i, B_j \in H$ and $a_i \geq r + \delta(r,q)$, $b_j \geq s + \delta(s,q)$, $\deg A_i = \deg R + \deg Q$, $\deg B_j = \deg S + \deg Q$. We may apply induction to the brackets $[A_i, S]$ and $[R, B_j]$ because $\deg A_i + \deg S = \deg R + \deg Q + \deg S = \deg P + \deg Q$ and $\deg R + \deg B_j = \deg R + \deg S + \deg Q = \deg P + \deg Q$, $\inf(A_i, S) > Q$ (because $S > Q$ and either $a_i > r \Rightarrow A_i > R > Q$ or $a_i = r$, hence $\delta(r,q) = 0 \Rightarrow r > q \Rightarrow a_i > q \Rightarrow A_i > Q$) and finally $\inf(R, B_j) > Q$ (because $R > Q$ and either $b_j > s \Rightarrow B_j > S > Q$ or $b_j = s$ hence $\delta(s,q) = 0 \Rightarrow s > q \Rightarrow b_j > q \Rightarrow B_j > Q$).

We obtain

$$[A_i, S] = \sum_k \gamma_{ik} C_k\,; \quad [R, B_j] = \sum_l \varepsilon_{jl} D_l\,;$$

where the C_k, D_l are in H and of degree $\deg P + \deg Q$, with $c_k \geq \sup(a_i, s) + \delta(a_i, s)$, $d_l \geq \sup(r, b_j) + \delta(r, b_j)$. As P is of level p, we have either (i) $r = s = p - 1$, or (ii) $r = p > s$. In the second case, we have $a_i \geq r$ hence $c_k \geq a_i \geq r = p$ and $d_l \geq r = p$. In the first case, either $a_i > r \Rightarrow c_k \geq a_i \geq r + 1 = p$, or $a_i \leq r \Rightarrow a_i = r = s \Rightarrow c_k \geq a_i + 1 = p$; moreover, either $b_j > s \Rightarrow d_l \geq b_j \geq s + 1 = p$, or $b_j \leq s \Rightarrow b_j = s = r \Rightarrow d_l \geq b_j + 1 = p$. In all cases, C_k and D_l are of level $\geq p = \sup(p,q) + \delta(p,q)$, which ends the induction step.

2.— We know by section 1 that H is a linearly independant set, hence it is enough to show that H_n generates $\mathcal{L}_n$. This is, for $n = 0$, simply the fact that $H_0 = H$ is a basis of $\mathcal{L}(A) = \mathcal{L}_0$. By bilinearity of the Lie bracket, and by induction, $\mathcal{L}_{n+1}$ is generated by the elements $[P, Q]$ with P, Q in H_n. By the first part of the proof, this implies that $\mathcal{L}_{n+1}$ is generated by the elements T in H with $\mathrm{l}(T) \geq \sup(p,q) + \delta(p,q)$, where $p = \mathrm{l}(P)$, $q = \mathrm{l}(Q)$, and $P, Q \in H_n$; then $\mathrm{l}(T) \geq n + 1$, hence $\mathcal{L}_{n+1}$ is generated by H_{n+1}. $\square$

The next lemma will be needed in the sequel of this paper. It gives a precise description of the Hall trees associated to a order on the free magma which is compatible with the level.

LEMMA 1. — *Let $n \geq 1$ and h a tree in $M(A)$. The following conditions are equivalent :*

(i) *h is a Hall tree of level n.*

(ii) *There exists $k \geq 2$ and k Hall trees $h_1, \ldots, h_k$ of level $(n-1)$ such that*

$$h = (\cdots((h_1, h_2), h_3), \ldots, h_k)$$

and $h_1 > h_2$, $h_2 \leq h_3 \leq \cdots \leq h_k$.

This result is directly inspired of Schützenberger's "décomposition normale gauche" of [6]. Figure 2 illustrates the LEMMA. Note that in particular, Hall trees of level 1 are precisely described : they correspond to the case where the h_i's are simply elements of A, with the inequality as in LEMMA 1.

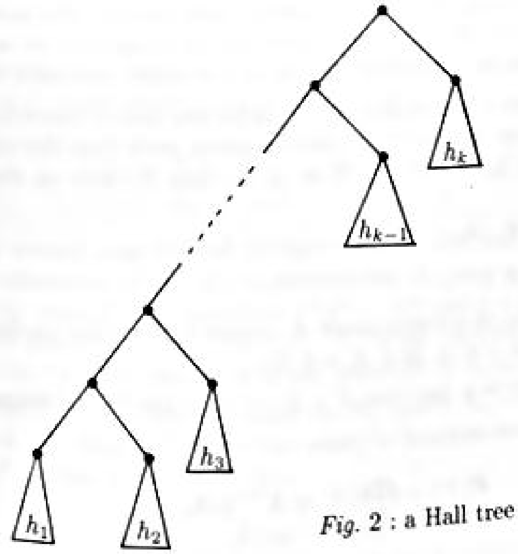

Fig. 2 : a Hall tree of level n

Proof. — (i) ⇒ (ii)

As h is of level ≥ 1, h is not in A. Hence $h = (h', h_k)$ where $h', h_k \in H$, $h' > h_k$ and either $h' \in A$, or $h' \in (u, v)$ with $v \leq h_k$. Then $\mathrm{l}(h') \geq \mathrm{l}(h_k)$ with equality only if $\mathrm{l}(h') = \mathrm{l}(h_k) = n - 1$; in this case, (ii) follows with $k = 2$ and $h_1 = h'$. If there is strict inequality, then $n = \mathrm{l}(h') > \mathrm{l}(h_k)$; by induction (on the number of leaves), there exists $k \geq 3$ and Hall trees $h_1, \dots, h_{k-1}$ of level $n - 1$ such that $h_1 > h_2$, $h_2 \leq \cdots \leq h_{k-1}$ and $h' = (\cdots(h_1, h_2), \dots, h_{k-1})$. Hence $h = ((\cdots(h_1, h_2), \dots, h_{k-1}), h_k) = ((u, h_{k-1}), h_k)$. By definition of Hall trees, $h_{k-1} \leq h_k$.

We know that $n > \mathrm{l}(h_k)$. If we had $n - 1 > \mathrm{l}(h_k)$, then $\mathrm{l}(h_k) < \mathrm{l}(h_{k-1}) \Rightarrow h_k < h_{k-1}$, which is a contradiction. Thus $n - 1 = \mathrm{l}(h_k)$ and (ii) follows.

(ii) ⇒ (i) (induction on k).

If $k = 2$, then either $h_1 \in A$, hence $\mathrm{l}(h_1) = 0 = \mathrm{l}(h_2)$ and h_2 is also in A, or $h_1 = (u, v)$, with v of level $n - 2$ by the first part of the proof; in this case $v < h_2$ because $\mathrm{l}(h_2) = n - 1$. In both cases, (h_1, h_2) is a Hall tree of level n.

If $k \geq 3$, then $h' = (\cdots(h_1, h_2), \dots, h_{k-1})$ is a Hall tree of level n, by induction on k; as $h = (h', h_k)$ with $\mathrm{l}(h') > \mathrm{l}(h_k)$ hence $h' > h_k$, and $h' = (h'', h_{k-1})$ with $h_{k-1} \leq h_k$, h is Hall tree of level n. □

3. Analytic functors

Analytic functors were introduced by A. Joyal [3], in relation with the combinatorial theory of species of structures. We need here the linear aspect of this notion, called "tensorial species" by Joyal. As noted by him, analytic functors are an extension of the polynomial functors of Macdonald [5] (Appendix to chapter 1). We shall recall here all the properties we need. The proofs may be found in the two cited papers, or are slight extension of these.

Let $\mathcal{V}$ be the category of vector spaces over $\mathbf{Q}$ (or any field of characteristic 0). Let (A_n) be a sequence of finite dimensional spaces, such that the symmetric group S_n acts on the left on A_n. If X is in $\mathcal{V}$, then S_n acts on the left on $X^{\otimes n}$ by

$$\sigma \cdot (x_1 \cdots x_n) = x_{\sigma^{-1}(1)} \cdots x_{\sigma^{-1}(n)}.$$

Hence S_n acts on $A_n \otimes X^{\otimes n}$.

For a group G acting on a vector space A, denote by A/G the quotient space of A by the relations $a = \sigma \cdot a$ $(a \in A, \sigma \in G)$.

An *analytic functor* is a function $F : \mathcal{V} \to \mathcal{V}$ of the form (defined up to natural isomorphism)

$$F(X) = \bigoplus_{n \geq 0} (A_n \otimes X^{\otimes n})/S_n \tag{5}$$

such that if $f : X \to Y$ is a linear mapping, then $F(f) : F(X) \to F(Y)$ is the linear mapping naturally induced by the mappings $\mathrm{id} \otimes f^{\otimes n} : A_n \otimes X^{\otimes n} \to A_n \otimes Y^{\otimes n}$.

Remark. — I. G. Macdonald [5] defines a polynomial functor to be a functor $F : \mathcal{V}_0 \to \mathcal{V}_0$ (where $\mathcal{V}_0$ is the category of finite dimensional vector spaces) such that the mapping $F : \mathrm{Hom}(X, Y) \mapsto \mathrm{Hom}(F(X), F(Y))$ is a polynomial mapping, for any X, Y in $\mathcal{V}_0$.

He shows that then there exists for each n an S_n-space A_n such that

$$F(X) = \bigoplus_{n \geq 0} (A_n \otimes X^{\otimes n})^{S_n}. \tag{6}$$

Now, there is a functorial isomorphism

$$A^G \longmapsto A/G$$

whenever G is a finite group acting on a vector space A (a variant of Maschke's theorem). Hence, each polynomial functor, in the sense of [5], is an analytic functor, in the sense of [3]. The terminology is somewhat misleading, because the polynomial case does not correspond to the case where the sum (5) is finite: actually, it may be shown that (5) is a polynomial functor in the sense of [5] exactly when for some N, the irreducible representations occurring in the A_n correspond to partitions whose parts are all $\leq N$.

It is useful to note that if F an analytic functor, then $F = \sum_n F_n$, where F_n is its homogeneous part of degree n, and each F_n is a polynomial functor (in the sense of Macdonald). Conversely, if for each n, F_n is a polynomial functor, then $\sum_n F_n$ is an analytic functor.

The representations A_n of S_n, called the *coefficients* of F, are unique. The *indicator* of F is the series in $\mathbf{Q}[[x_1, x_2, \cdots, x_k, \cdots]]$

$$Z_F = \sum_{n \geq 0} \frac{1}{n!} \sum_{\sigma \in S_n} \chi_n(\sigma) x_\sigma = \sum_I \frac{\chi(I)}{\mathrm{aut}(I)}$$

where the second sum is over all partitions I, with the following notation: χ_n is the character of the S_n-representation A_n; $x_\sigma = x_I$ is the monomial $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ when I is the partition $1^{i_1} 2^{i_2} \dots n^{i_n}$ and σ a permutation of cycle-type I, and $\mathrm{aut}(I) = 1^{i_1} 2^{i_2} \cdots n^{i_n} i_1! \cdots i_n!$ is the cardinality of the centralizer of σ; χ is the "global" character of the sequence of S_n-spaces (A_n).

Note that Z_F determines uniquely the S_n-spaces A_n, hence F. A fundamental result of [3] is that if F, G are two analytic functors such that the "constant term" of F is zero, *i.e.* $A_0 = 0$, then

$$Z_{G \circ F} = Z_G(Z_F)$$

where the right member is *phethystic substitution*, that is,

$$Z_G(Z_F) = Z_G(Z_1, Z_2, \ldots, Z_k, \ldots)$$

with $Z_i = Z_F(x_i, x_{2i}, \ldots, x_{ki}, \ldots)$.

The *generating series* of an analytic functor is the one-variable series

$$g_F = \sum_{n \geq 0} \frac{\dim A_n}{n!} x^n.$$

Hence $g_F = Z_F(x, 0, \ldots, 0, \ldots)$.

The *characteristic symmetric function* $\chi(F)$ of F is defined in the following way : take X a space with a denumerable basis $a_1, a_2, \ldots, a_n, \ldots$; then this basis defines a multi-graduation of each $X^{\otimes n}$, which is inherited by $F(X)$. Denote by $\alpha_{(i)}$ the dimension of the homogeneous component of $F(X)$ of multidegree $(i) = (i_2, i_2, \ldots, i_k, \ldots)$. Then the characteristic symmetric function is

$$\chi(F) = \sum_{(i)=(i_1,\ldots,i_k,\ldots)} \alpha_{(i)} x_1^{i_1} \cdots x_k^{i_k} \cdots$$

If A_n is decomposed into irreducible representations (with a_I = the multiplicity of the irreducible representation corresponding to the partition I) then

$$\chi(F) = \sum_I a_I s_I$$

where s_I is the Schur function corresponding to I. Another important formula relates characteristic function and indicator :

$$\chi(F) = Z_F(p_1, p_2, \ldots, p_k, \ldots)$$

where p_n is the power sum $p_n = \sum_i x_i^n$.

Examples of analytic functors are given below.

1. The tensor algebra

$$T(X) = \bigoplus_{n \geq 0} X^{\otimes n}.$$

Here the coefficients are the regular representations of the symmetric groups. The indicator is $1/(1 - x_1)$ and the characteristic function is $(1 - \sum_i x_i)^{-1}$.

2. The symmetric algebra

$$S(X) = \bigoplus_{n \geq 0} (\mathbf{Q} \otimes X^{\otimes n})/S_n,$$

where S_n acts trivially on $\mathbf{Q}$, *i.e.* the coefficients are the trivial representations. The indicator is $\exp(\sum_{i \geq 1} x_i/i) = \sum x_I/\mathrm{aut}(I) = \sum_{n \geq 0} \sum_{\sigma \in S_n} x_\sigma/n!$ and the characteristic function is $\prod_{i \geq 1} 1/(1 - x_i)$.

3. The exterior algebra

$$\bigwedge(X) = \bigoplus_{n \geq 0} (\varepsilon_n \otimes X^{\otimes n})/S_n$$

where the coefficients ε_n are the alternating representations. The indicator is $\exp(\sum_{i \geq 1} (-1)^{i-1} x_i/i)$ and the characteristic function is $\prod_{i \geq 1}(1 + x_i)$.

4. (Joyal [3]) The free Lie algebra on the vector space X

$$L(X) = \bigoplus_{n \geq 0} (L_n \otimes X^{\otimes n})/S_n$$

with indicator series

$$\sum_{1 \leq d | n} \frac{\mu(d)}{n} x_d^{n/d}$$

(with μ = the Mobius function) and characteristic function

$$\sum_{(i)=(i_1,\ldots,i_k,\cdots)} \alpha_{(i)} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} \cdots$$

where the α's are defined by

$$\frac{1}{1 - \sum_{i \geq 1} x_i} = \prod_{(i)} \Big(\frac{1}{1 - x_1^{i_1} \cdots x_k^{i_k} \cdots} \Big)^{\alpha_{(i)}}.$$

Note that in these examples, only the exterior algebra is a polynomial functor in the sense of Macdonald :

- $\bigwedge(X)$ is finite-dimensional for any finite-dimensional X, but
- $T(X)$, $S(X)$ and $L(X)$ have infinite dimension as soon as $X \neq 0$.

An important class of analytic functors will be described now (it contains all the previous examples). Let $A = \{a_1, \ldots, a_n, \ldots\}$ be an infinite alphabet and $\mathcal{F}$ a subspace of $\mathbf{Q}\langle A \rangle$ such that

$$(*) \qquad \left\{ \begin{array}{l} \textit{For any algebra endomorphism } \phi \textit{ of } \mathbf{Q}\langle A \rangle \textit{ satisfying} \\ \forall a \in A, \ \phi(a) \textit{ is homogeneous of degree } 1, \\ \textit{one has } \phi(\mathcal{F}) \subseteq \mathcal{F}. \end{array} \right.$$

We define a functor in the following way : first let X be a finite-dimensional space; then there exists a mapping $\alpha : A \to X$ such that $\alpha(A)$ generates X.

There is a unique extension of α to a mapping $\alpha : \mathbf{Q}\langle A\rangle \to T(X)$, and we define $F(X) = \alpha(\mathcal{F})$.

Because of (*), $\alpha(\mathcal{F})$ does not depend on α subject to the above generating condition. If X is an artitrary vector space, it is the inductive limit of its finite dimensional subspaces X_i; then $F(X)$ is by definition the inductive limit of the subspaces $F(X_i)$ (it is naturally embedded in $T(X)$).

The functor F defined in this way is analytic; this may be verified as follows : write $\mathcal{F} = \bigotimes_n \mathcal{F}_n$, where each $\mathcal{F}_n$ is the homogeneous component of degree n of $\mathcal{F}$, with the usual degree on $\mathbf{Q}\langle A\rangle$. To each $\mathcal{F}_n$ corresponds a functor F_n and we have $F = \sum_n F_n$. Each functor F_n is polynomial in the sense of MACDONALD, because the induced mapping $F_n : \mathrm{Hom}(X,Y) \to \mathrm{Hom}(F_n(X), F_n(Y))$ is polynomial for any finite-dimensional spaces X, Y (see the remark). Now, as F_n preserves inductive limits, F_n is analytic and F, as a sum of homogeneous polynomial functors, is analytic.

The coefficients of F are easily described : let A_n be the subspace $\mathcal{F} \cap E_n$, where E_n is generated by the $n!$ words $a_{\sigma(1)}, \ldots, a_{\sigma(n)}$, $\sigma \in S_n$. Then A_n is naturally an S_n-space (because of (*)), and these A_n are the coefficients of F.

Similarly, the characteristic function of F may be described in the following way : for a multi-index $(i) = (i_1, i_2, \ldots, i_k, \ldots)$, let $E_{(i)}$ be the subspace of $\mathbf{Q}\langle A\rangle$ formed by the homogeneous polynomials of multidegree (i). Define $\alpha_{(i)} = \dim(E_{(i)} \cap \mathcal{F})$.

Then

$$\chi(F) = \sum_{(i)} \alpha_{(i)} x_1^{i_1} \cdots x_k^{i_k} \cdots$$

We need to generalize slightly this class of anatytic functors. Let $\mathcal{G}$, $\mathcal{F}$ be subspaces of $\mathbf{Q}\langle A\rangle$ both satisfying (*), and such that $\mathcal{G} \subset \mathcal{F}$. Then one has a natural embedding $G(X) \subset F(X)$, for any space X. Then the functor

$$X \longmapsto F(X)/G(X)$$

is analytic (similar proof as above). Its coefficients are the S_n-spaces A_n/B_n (where A_n, B_n are the coefficients of F, G) and its symmetric function is $\chi(F) - \chi(G)$.

4. – Dimensions and characters of the derived series

We apply the previous section to the situation of section 2.

Let $A = \{a_1, \cdots, a_n, \cdots\}$ as in section 3 and define $\mathcal{F}_n = \mathcal{L}_n(A)$ to be the n-th term of the derived series of the free Lie algebra $\mathcal{L}(A)$. It is clear that the condition (*) of section 3 is satisfied, because the homomorphism φ of (*) sends the Lie algebra into itself. Hence we obtain an analytic functor $X \mapsto L_n(X)$, which is the n-th term of the free Lie algebra over the space X.

In particular, $L_0(X)$ is the free Lie algebra, whose indicator is given in section 3. Its generating series is simply

$$g_{L_0} = \sum \frac{x^n}{n} = \log\left(\frac{1}{1-x}\right)$$

As we have $\mathcal{L}_{n+1} \subset \mathcal{L}_n$, we may define, using section 3, an analytic functor

$$F_n(X) = L_n(X)/L_{n+1}(X).$$

It is clear that $F_0(X) = X$, *i.e* F_0 is the identity. There exists a remarkable connection between F_1 and F_n, which is our main result.

THÉORÈME 2.

(i) *The indicator of* F_1 *is*

$$Z_{F_1} = 1 + (x_1 - 1)\exp\Big(\sum_{i \geq 1} \frac{x_i}{i}\Big).$$

Its characteristic function is

$$\chi(F_1) = \sum_{n \geq 2} s_{1,n-1},$$

the sum of all Schur functions corresponding to partitions $(1, n-1)$. *Its generating series is*

$$g_{F_1} = 1 + (x-1)e^x = \sum_{n \geq 2} \frac{n-1}{n!} x^n.$$

(ii) *The functor* F_n *is equal to the n-th composition of* F_1 *with itself*

$$F_n = F_1^{(n)}.$$

Equivalently, the indicator (resp. the characteristic function) F_n *is the n-th plethystic substitution of* Z_{F_1} *(resp.* $\chi(F_1)$*) into itsef. In particular,* $g_{F_n} = g_{F_1}^{(n)}$.

Proof.

(i) A basis of $\mathcal{L}_1$ mod $\mathcal{L}_2$ is formed by all the Hall polynomials

$$[[a_{i_1}, a_{i_2}], a_{i_3}], \cdots], a_{i_k}]$$

with $k \geq 2$, $i_1 > i_2 \leq i_3 \leq \cdots \leq i_k$ (see THÉORÈME 1 and LEMMA 1; we order A naturally). All these Hall polynomials are homogeneous with respect to all partial degrees. By the description in section 3 of $\chi(F_1)$, we thus obtain that

$$\chi(F_1) = \sum x_{i_1} x_{i_2} \cdots x_{i_k},$$

where the sum is extended to all i_j satisfying the above condition. But this condition simply says that

$$\begin{array}{llll} i_1 & & & \\ i_2 & i_2 & \dots & i_k \end{array}$$

is a Young tableau. Hence, by [5] (I.5.1.2), we obtain that $\chi(F_1)$ is the sum of all Schur functions corresponding to the shapes $(1, n-1)$, $n \geq 2$. In order to obtain the indicator, we have just to compute the corresponding characters of S_n. But it is well-known that the natural representation of S_n is for $n \geq 2$ the sum of the trivial one, and the one corresponding to the partition $(1, n-1)$; call the latter A_n (and these A_n are the coefficients of F_1, by the beginning of the proof). Let N denote the functor whose coefficients are the natural representations of S_n; recall that $S(X)$ denotes the symmetric algebra, the functor whose coefficients are the trivial representations of S_n. We have seen that $1 + N = F_1 + S$ hence $1 + Z_N = Z_{F_1} + Z_S$ (the correcting term 1 corresponds to the symmetric group S_0).

The indicator of S is

$$Z_S = \exp\Big(\sum_{i \geq 1} \frac{x_i}{i}\Big) = \sum_{n \geq 0} \sum_{\sigma \in S_n} \frac{x_\sigma}{n!}.$$

But the indicator of N is

$$Z_N = \sum_{n \geq 0} \sum_{\sigma \in S_n} \alpha_1(\sigma) \frac{x_\sigma}{n!}$$

where $\alpha_1(\sigma)$ is the number of fixed points of σ, which is the exponent of x_1 in x_σ. Hence

$$Z_N = x_1 \frac{d}{dx_1}(Z_S) = x_1 \exp\Big(\sum_{i \geq 1} \frac{x_i}{i}\Big).$$

Thus we have

$$Z_{F_1} = 1 + Z_N - Z_S = 1 + (x_1 - 1) \exp\Big(\sum_{i \geq 1} \frac{x_i}{i}\Big).$$

The generating function of F_1 is obtained by putting $x_1 = x, x_2 = x_3 = \cdots = 0$.

(ii) We have just to show that

$$\chi(F_{n+1}) = \chi(F_1)\big[\chi(F_n)\big],$$

where the brackets mean plethysm of symmetric functions. By THEOREM 1, $\chi(F_n)$ is the sum of all commutative images of Hall trees of level 1. This together with LEMMA 1 immediately implies the last relation, once is recalled the plethysm of symmetric function : if $S(x_1, \dots, x_n, \dots)$ and T are symmetric functions with nonnegative coefficients, then write $T = \sum_{i=1}^{\infty} m_i$ for monomials m_i. Then the plethysm $S[T]$ of T in S is the symmetric function $S(m_1, m_2, \dots)$. ☐

COROLLAIRE. — *The functor "free Lie algebra" is the sum of all the substitution powers of the functor $F_1(X) = L_1(X)/L_2(X)$.*

Proof. — There is an isomorphism of vector spaces

$$\mathcal{L}(A) \simeq \bigoplus_{n \geq 0} \mathcal{L}_n(A)/\mathcal{L}_{n+1}(A)$$

which implies a functorial decomposition

$$L(X) = \sum_{n \geq 0} F_n(X) = \sum_{n \geq 0} (F_1^{(n)}(X))$$

(the last equality by th. 2). ☐

Remark.

One obtains similar results for the indicator and characteristic function of the free Lie algebra. In particular, its generating series satisfies the rather curious relation

$$g_L = \log\Big(\frac{1}{1-x}\Big) = \sum_{n \geq 0} g_{F_1}^{(n)}$$

and $g_{F_1} = \sum_{k \geq 2}((k-1)/k!)x^k = 1 + (x-1)e^x$. P. LEROUX pointed out that this last relation is equivalent to say that $y = \log(1-x)^{-1}$ satisfies the functional equation

$$\begin{aligned} y(x) &= x + y\Big(\sum_{n \geq 2} \frac{n-1}{n!} x^n\Big) \\ &= x + y\big(1 + (x-1)e^x\big), \end{aligned}$$

which is indeed true, as is immediately verified. Similarly, $L(X)$ satisfies the functional equation

$$L(X) = X + L(F_1(X)).$$

See [1] for similar functional equations in the theory of species.

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